The De Pril transform of a compound $\mathcal{R}_k$ distribution

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Abstract

In this paper we present a recursion for the De Pril transform of a compound distribution whose counting distribution belongs to the class $\mathcal{R}_k$, and discuss application of this recursion for evaluation of convolutions of compound distributions. Special emphasis will be put on evaluation of the aggregate claims distribution of a heterogeneous portfolio of independent policies where the aggregate claims distribution of each policy can be represented as a compound negative binomial distribution.

1 Introduction

1A. The De Pril transform was defined by Sundt (1995) for probability distributions on the non-negative integers with a positive probability at zero. He discussed several properties of the De Pril transform. In particular, he derived a recursion for the De Pril transform of distributions in the classes $\mathcal{R}_k$ studied by Sundt (1992). He also derived an expression for the De Pril transform of a compound distribution expressed by convolutions of the severity distribution and the De Pril transform of the counting distribution. Unfortunately, evaluation of this expression would normally be rather time-consuming. In the present paper we shall consider the special case when the counting distribution belongs to $\mathcal{R}_k$. By applying the recursion for De Pril transforms of distributions in $\mathcal{R}_k$ and a recursion derived by Sundt (1992) for a compound distribution with counting distribution in $\mathcal{R}_k$, we shall deduce a recursion for the De Pril transform of the compound distribution. When k is small, this

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recursion could be less time-consuming than the expression given by Sundt (1995). This also implies that in this case one would be less inclined to apply approximations instead of exact evaluation.

1B. In Section 2 we briefly recapitulate some results on the De Pril transform and the classes $\mathcal{R}_k$ from Sundt (1992, 1995) and present our new recursion. In Section 3 we discuss this recursion in the special case when the counting distribution belongs to $\mathcal{R}_1$. Section 4 is devoted to the question whether to apply exact or approximate evaluation of convolutions of compound distributions. This question is further discussed in Section 5 in a special case where we want to evaluate the aggregate claims distribution of a heterogeneous portfolio.

1C. In the present paper we shall represent a distribution by its probability function, and hence we shall normally mean its probability function when talking about a distribution.

2 General theory

2A. We shall say that a probability distribution on the non-negative integers with a positive mass at zero is $R_k[a, b]$ if there exist functions a and b such that its probability function p satisfies the recursion

$$p(n) = \sum_{i=1}^{k} \left(a(i) + \frac{b(i)}{n} \right) p(n-i) \qquad (n = 1, 2, \ldots)$$

with $p(n) = 0$ for all $n < 0$. For simplicity we shall always let $a(i) = b(i) = 0$ for all $i > k$. We shall call the class of all such distributions with a fixed k $\mathcal{R}_k$. These classes were introduced by Sundt (1992), who discussed several of their properties.

We obviously have that $\mathcal{R}_{k-1} \subset \mathcal{R}_k$ for all k. The class $\mathcal{R}_\infty$ consists of all distributions on the non-negative integers with a positive mass at zero.

2B. For any distribution $f \in \mathcal{R}_\infty$, there exists a unique function φ_f on the positive integers such that f can be represented as $R_\infty \left[0, \varphi_f\right]$. This is seen by solving the recursion

$$f(x) = \frac{1}{x} \sum_{y=1}^{x} \varphi_f(y) f(x-y) \qquad (x = 1, 2, \ldots) \tag{2.1}$$

for $\varphi_f(x)$ to obtain

$$\varphi_f(x) = \frac{1}{f(0)} \left(x f(x) - \sum_{y=1}^{x-1} \varphi_f(y) f(x-y) \right); \qquad (x = 1, 2, \ldots) \tag{2.2}$$

in this paper we shall interpret $\sum_{i=s}^{t} v_i = 0$ when $s > t$. Sundt (1995) called φ_f the *De Pril transform* of f. He studied its properties and argued that it can be a useful tool for recursive evaluation of distributions in $\mathcal{R}_\infty$.

The De Pril transform is additive in the sense that if $f_1, f_2, \ldots, f_m \in \mathcal{R}_\infty$, then

$$\varphi_{*_{j=1}^{m} f_j} = \sum_{j=1}^{m} \varphi_{f_j}. \tag{2.3}$$

Sundt (1995) showed that if p is $R_k[a,b]$, then

$$\varphi_p(n) = na(n) + b(n) + \sum_{i=1}^{k} a(i)\,\varphi_p(n-i) \qquad (n = 1, 2, \ldots) \tag{2.4}$$

with $\varphi_p(n) = 0$ for all negative n.

2C. Let $\mathcal{P}_+$ denote the class of distributions on the positive integers. The compound distribution $p \vee h$ with counting distribution on the non-negative integers and severity distribution $h \in \mathcal{P}_+$ is given by

$$(p \vee h)(x) = \sum_{n=0}^{x} p(n)\,h^{n*}(x). \qquad (x = 0, 1, 2, \ldots)$$

We immediately see that if $p \in \mathcal{R}_\infty$, then $p \vee h \in \mathcal{R}_\infty$.

Sundt (1992) showed that if p is $R_k[a,b]$, then $p \vee h$ is $R_\infty[c,d]$ with

$$c(x) = \sum_{y=1}^{k} a(y)\,h^{y*}(x); \qquad d(x) = x\sum_{y=1}^{k} \frac{b(y)}{y} h^{y*}(x). \qquad (x = 1, 2, \ldots) \tag{2.5}$$

In particular, with $k = \infty$, $a = 0$, and $b = \varphi_p$, we obtain that for every $p \in \mathcal{R}_\infty$ we have that

$$\varphi_{p\vee h}(x) = x\sum_{y=1}^{x} \frac{\varphi_p(y)}{y} h^{y*}(x) \qquad (x = 1, 2, \ldots) \tag{2.6}$$

as $h^{y*}(x) = 0$ for all $y > x$. The relation (2.6) was shown by Sundt (1995).

2D. Let us now assume that p is $R_k[a,b]$. By (2.6) with c and d given by (2.5) we obtain that for $x = 1, 2, \ldots$

$$\varphi_{p\vee h}(x) = xc(x) + d(x) + \sum_{y=1}^{x-1} c(y)\,\varphi_{p\vee h}(x-y),$$

that is,

$$\varphi_{p\vee h}(x) = x\sum_{y=1}^{k}\left(a(y) + \frac{b(y)}{y}\right) h^{y*}(x) + \sum_{y=1}^{x-1} \varphi_{p\vee h}(x-y) \sum_{z=1}^{k} a(z)\,h^{z*}(y). \tag{2.7}$$

By letting $k=\infty$, $a=0$, and $b=\varphi_p$ in (2.7) we obtain (2.6).

We see that in particular for low values of k (2.7) is much less time-consuming than (2.6) as in (2.7) one would need h^{i*} only for $i=1,\ldots,k$.

3 The case $k=1$

3A. Let us now consider the special case $k=1$. Then (2.4) gives

$$\varphi_p(n)=(a+b)\,a^{n-1}. \qquad (x=1,2,\ldots) \tag{3.1}$$

Furthermore, (2.7) reduces to

$$\varphi_{p\vee h}(x)=(a+b)\,xh(x)+a\sum_{y=1}^{x-1}h(y)\,\varphi_{p\vee h}(x-y)\,, \qquad (x=1,2,\ldots) \tag{3.2}$$

whereas insertion of (3.1) in (2.6) gives

$$\varphi_{p\vee h}(x)=x\,(a+b)\sum_{y=1}^{x}\frac{a^{y-1}}{y}h^{y*}(x)\,. \qquad (x=1,2,\ldots) \tag{3.3}$$

It is well known that $R_1[a,b]$ is binomial if $a<0$, Poisson if $a=0$, and negative binomial if $a>0$ (cf. Sundt & Jewell (1981)). Let us consider (3.1) and (3.2) in each of these three cases.

3B. *Binomial with parameters* (t,π).

$$p(n)=\binom{t}{n}\pi^n(1-\pi)^{t-n}\,. \qquad (n=0,1,\ldots,t;\ t=1,2,\ldots;\ 0<\pi<1)$$

Then

$$a=-\frac{\pi}{1-\pi}; \qquad b=(t+1)\frac{\pi}{1-\pi}$$

$$\varphi_p(n)=-t\left(\frac{\pi}{\pi-1}\right)^n \qquad (n=1,2,\ldots) \tag{3.4}$$

$$\varphi_{p\vee h}(x)=\frac{\pi}{1-\pi}\left(txh(x)-\sum_{y=1}^{x-1}h(y)\,\varphi_{p\vee h}(x-y)\right). \qquad (x=1,2,\ldots) \tag{3.5}$$

Any distribution $f\in\mathcal{R}_\infty$ can be represented in the form $p\vee h$ where p is a Bernoulli distribution with parameter

$$\pi=1-p(0)\,,$$

(that is, a binomial distribution with parameters $(1, \pi)$) and $h \in \mathcal{P}_+$ is given by

$$h(x) = \frac{f(x)}{\pi}. \qquad (x = 1, 2, \ldots)$$

Insertion in (2.2) gives

$$\varphi_f(x) = \frac{\pi}{1-\pi}\left(xh(x) - \sum_{y=1}^{x-1} h(y)\varphi_f(x-y)\right), \qquad (x = 1, 2, \ldots)$$

that is, we obtain (3.5) with $t = 1$.

In general we can write (3.5) as

$$\frac{\varphi_{p\vee h}(x)}{t} = \frac{\pi}{1-\pi}\left(xh(x) - \sum_{y=1}^{x-1} h(y)\frac{\varphi_{p\vee h}(x-y)}{t}\right), \qquad (x = 1, 2, \ldots)$$

which immediately follows from the Bernoulli case and (2.3) as the compound binomial distribution is the t-fold convolution of the corresponding compound Bernoulli distribution.

In the Bernoulli case with $t = 1$, (3.5) was given in formula (2) in De Pril (1989).

3C. *Poisson distribution with parameter* λ.

$$p(n) = \frac{\lambda^n}{n!}e^{-\lambda}. \qquad (n = 0, 1, \ldots;\ \lambda > 0)$$

Then

$$a = 0; \qquad b = \lambda \tag{3.6}$$

$$\varphi_p(n) = \begin{cases} \lambda & (n = 1) \\ 0 & (n = 2, 3, \ldots) \end{cases} \tag{3.7}$$

$$\varphi_{p\vee h}(x) = \lambda x h(x); \qquad (x = 1, 2, \ldots) \tag{3.8}$$

the latter formula is also obtained by inserting (3.6) in (3.3).

3D. *Negative binomial distribution with parameters* (α, π).

$$p(n) = \binom{\alpha + n - 1}{n}(1-\pi)^\alpha \pi^n. \qquad (n = 0, 1, \ldots;\ \alpha > 0;\ 0 < \pi < 1)$$

Then

$$a = \pi; \qquad b = (\alpha - 1)\pi \tag{3.9}$$

$$\varphi_p(n) = \alpha\pi^n \qquad (n = 1, 2, \ldots)$$

$$\varphi_{p\vee h}(x) = \pi\left(\alpha x h(x) + \sum_{y=1}^{x-1} h(y)\,\varphi_{p\vee h}(x-y)\right). \qquad (x = 1, 2, \ldots) \quad (3.10)$$

Insertion of

$$\lambda = -\ln p(0) = \alpha\,|\ln(1-\pi)|$$

$$k(x) = \frac{\varphi_{p\vee h}(x)}{\lambda x} \qquad (x = 1, 2, \ldots) \quad (3.11)$$

in (3.10) and division by λx gives

$$k(x) = \pi\left(\frac{h(x)}{|\ln(1-\pi)|} + \sum_{y=1}^{x-1}\left(1 - \frac{y}{x}\right) h(y)\,k(x-y)\right). \qquad (x = 1, 2, \ldots)$$

This is the recursion presented by Sundt & Jewell (1981) (apart from a misprint in that paper) for the compound distribution $q \vee h$, where q is the logarithmic distribution given by

$$q(x) = \frac{1}{|\ln(1-\pi)|}\frac{\pi^x}{x}. \qquad (x = 1, 2, \ldots)$$

Thus $k = q \vee h$. From (3.8) and (3.11) we conclude that $p \vee h = r \vee k$, where r is the Poisson distribution with parameter λ. Thus

$$p \vee h = r \vee (q \vee h) = (r \vee q) \vee h.$$

In particular, when h is concentrated in one, we obtain that $p = r \vee q$. This representation of a negative binomial distribution as a compound Poisson distribution with a logarithmic severity distribution was presented independently by Ammeter (1949) and Quenouille (1949).

3E. To summarise, in the binomial case the recursion (3.2) gives simply a reformulation of the general recursion (2.2) for the De Pril transform, in the Poisson case it is trivial, and in the negative binomial case the recursion has earlier been deduced within another context. Thus, with counting distributions in $\mathcal{R}_1$ (2.7) does not bring much new. However, it gives a unification.

4 Exact evaluation or approximation?

4A. To cover approximations to distributions when the approximations are not necessarily distributions themselves, Dhaene & Sundt (1998) extended the definition of the De Pril transform to the class $\mathcal{F}_0$ of functions on the non-negative integers with a positive mass at zero. Formula (2.7) is easily generalised to that situation.

4B. By combining (2.3) and (2.6) we obtain that if $p_j \in \mathcal{R}_\infty$ and $h_j \in \mathcal{P}_+$, then

$$\varphi_{*_{j=1}^{m}(p_j \vee h_j)}(x) = \sum_{j=1}^{m} \varphi_{p_j \vee h_j}(x) = x \sum_{y=1}^{x} \frac{1}{y} \sum_{j=1}^{m} \varphi_{p_j}(y)\, h_j^{y*}(x) \,. \qquad (x = 1, 2, \ldots) \tag{4.1}$$

When x gets large, evaluation of this formula gets rather time-consuming. Therefore Dhaene & Sundt (1998) suggested as an approximation to replace $\varphi_{p_j}(y)$ by zero when y is greater than some integer r. They also discussed error bounds for such approximations. Such approximations are in particular interesting when $\varphi_{p_j}(y)$ rapidly approaches zero when y increases. Such approximations were studied by De Pril (1989) in the special case when the p_j's are Bernoulli distributions (cf. subsection 3B), and we shall therefore call them *De Pril approximations.*

When deciding whether to apply an approximation or an exact method, computation time should be considered against the need for accuracy. Analogous considerations would be needed when deciding the order r of the approximation.

As pointed out at the end of subsection 2D, (2.7) could be much less time-consuming than (2.6) when $p_j \in \mathcal{R}_k$ with a small k. Thus, in that case (2.7) would make exact evaluation more attractive.

4C. Dhaene & Sundt (1998) in particular discussed the case when p_j is $R_1[a_j, b_j]$. Obviously the approximation is interesting only when $a_j > 0$, that is, when p_j is either binomial or negative binomial. However, we also do not want a_j to be too far from zero as we want $\varphi_{p_j}(y)$ to rapidly approach zero when y increases.

Sundt & Jewell (1981) showed that we always have $a_j < 1$. For $a_j \leq -1$, $\varphi_{p_j}(y)$ diverges when $y \uparrow \infty$. >From the discussion on $\mathcal{R}_1$ in Section 3 follows that that occurs only in the binomial case with $\pi_j \geq 1/2$.

Not surprisingly, the error bounds discussed by Dhaene & Sundt (1998) increase when $|a_j|$ increases. When $|a_j| \uparrow 1$, the error bounds go to infinity.

The Bernoulli case has been discussed by De Pril (1989) in a situation where we consider m independent policies over a specified period. For $j = 1, \ldots, m$, we let π_j denote the probability that the aggregate claim amount of policy j is positive and h_j the conditional distribution of the aggregate claim amount of the policy given that the aggregate claim amount is positive. Then the unconditional aggregate claims distribution of the policy is the compound distribution $p_j \vee h_j$, where p_j denotes the Bernoulli distribution with parameter π_j. The De Pril approximation seems reasonable when the probabilities of non-zero claims are small.

In the following section we shall discuss an application of the negative binomial case.

5 Modelling heterogeneous portfolios

5A. We want to evaluate the aggregate claims distribution f of an insurance portfolio of m independent policies over a specified period. For the moment we assume that the aggregate claim amount for policy j $(j = 1, \ldots, m)$ has the compound distribution $p_j \vee h_j$, where $h_j \in \mathcal{P}_+$ and p_j is Poisson with Poisson parameter λ_j. From Theorem 10.1 in Sundt (1993) it follows that

$$f = \mathop{*}_{j=1}^{m} (p_j \vee h_j) = p \vee h,$$

where p is Poisson with parameter

$$\lambda = \sum_{j=1}^{m} \lambda_j$$

and

$$h = \frac{1}{\lambda} \sum_{j=1}^{m} \lambda_j h_j.$$

Thus we can evaluate f by the Panjer (1980) recursion

$$f(x) = \frac{\lambda}{x} \sum_{y=1}^{x} y h(y) f(x-y), \qquad (x = 1, 2, \ldots)$$

which easily follows by insertion of (3.8) in (2.1).

5B. Often one would expect that there are individual properties of an insurance policy that affect the risk, but are not reflected by the objective rating criteria applied. We shall assume that these properties affect only the claim numbers, not the severities. We assume that to each policy j there is related a positive random variable Θ_j, and that $\Theta_1, \ldots, \Theta_m$ are independent and identically distributed. It is assumed that the conditional distribution of the number of claims from policy j given that $\Theta_j = \theta$, is Poisson with parameter $\theta\lambda_j$. Thus the unconditional distribution p_j is given by

$$p_j(n) = \frac{\lambda_j^n}{n!} \mathrm{E}\, \Theta_j^n e^{-\Theta_j \lambda}. \qquad (n = 0, 1, \ldots) \tag{5.1}$$

Let us assume that the Θ_j's are gamma distributed with density

$$u(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta\theta}. \qquad (\theta > 0;\ \alpha, \beta > 0) \tag{5.2}$$

Then we easily get

$$\mathrm{E}\,\Theta_j^n e^{-\Theta_j \lambda} = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} \frac{\beta^\alpha}{(\beta+\lambda_j)^{\alpha+n}}, \qquad (n = 0, 1, \ldots)$$

and by insertion in (5.1) and some manipulation we obtain

$$p_j(n) = \binom{\alpha+n-1}{n} \left(\frac{\beta}{\beta+\lambda_j}\right)^\alpha \left(\frac{\lambda_j}{\beta+\lambda_j}\right)^n, \qquad (n = 0, 1, \ldots)$$

that is, p_j is negative binomial with parameters $\left(\alpha, \dfrac{\lambda_j}{\beta+\lambda_j}\right)$. By (3.9) this implies that p_j is $R_1[a_j, b_j]$ with

$$a_j = \frac{\lambda_j}{\beta+\lambda_j}; \qquad b_j = (\alpha-1)\frac{\lambda_j}{\beta+\lambda_j}. \tag{5.3}$$

5C. In the situation of subsection 5A it was fairly easy to evaluate the aggregate claims distribution f of the portfolio. As the policies were independent and the aggregate claims distribution of each policy was compound Poisson, also f is compound Poisson. Unfortunately it is not that simple in the negative binomial case of subsection 5B. In the restrictive case when λ_j and h_j are independent of j for all j, f would be a compound negative binomial distribution; in the general case, f would usually not be a compound negative binomial distribution. One possibility would then be to for each j evaluate $p_j \vee h_j$ by the Panjer (1981) recursion

$$(p_j \vee h_j)(x) = \sum_{y=1}^{x} \left(a_j + b_j \frac{y}{x}\right) h_j(y) (p_j \vee h_j)(x-y), \quad (x = 1, 2, \ldots)$$

and then find $f = *_{j=1}^m (p_j \vee h_j)$ by brute force convolutions. However, it seems more efficient to evaluate the De Pril transform of f either exact or approximate, and then find f by the recursion (2.1).

For approximate evaluation one could apply the De Pril approximation in (4.1). As argued in subsection 4B, this approximation seems reasonable if a_j is small, that is, by (5.3) when λ_j is small. This could be interpreted as if policy j has low risk exposure, that is, a small policy.

For large policies it seems more appropriate to apply exact evaluation. In that case one could for each of the policies evaluate the De Pril transform by the recursion (3.10) and then sum these De Pril transforms to obtain the De Pril transform of f. Numerical evaluation applying this methodology on data from group life assurance have been carried out by Ekuma (1998). This methodology is closely related to the methodology discussed in Section 5 of Willmot & Sundt (1989).

5D. The parameter λ_j can be interpreted as a measure of the risk volume of policy j. This interpretation becomes perhaps most clear when considering group insurances. Let us look at a simplified example. We consider a group life assurance portfolio. Policy j covers the employees of firm j. At any time during the period that firm has n_j employees (that is, we assume that if an employee dies during the period, he is immediately replaced with another). Conditional on $\Theta_j = \theta$, the lives of these employees are independent, and the conditional mortality rate of each employee is $\theta\mu$. Then the conditions of subsection 5B are fulfilled with $\lambda_j = n_j\mu$, and we see that a large value of λ_j means a firm with many employees.

5E. In the present situation we have for simplicity considered the unconditional distribution of the aggregate claims distribution of the portfolio. However, it would be natural to believe that the number of claims of policy j from earlier years would contain information about Θ_j, so that one should rather apply the conditional distribution given the claim experience of the individual policies. As the gamma distributions constitute a conjugate class to the Poisson distribution, one would under reasonable assumptions obtain that also the conditional distribution of Θ_j given the claim experience is gamma (cf. e.g. Section 2 in Norberg (1989)). Thus the conditional claim number distribution of policy j is negative binomial, and the discussion of subsection 5C is still valid.

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References

Ammeter, H. (1949). Die Elemente der kollektiven Risikotheorie von festen und zufallsartig schwankenden Grundwahrscheinlichkeiten. *Mitteilungen der Vereinigung schweizerischer Versicherungsmathematiker* **49**, 35-95.

De Pril, N. (1989). The aggregate claims distribution in the individual model with arbitrary positive claims. *ASTIN Bulletin* **19**, 9-24.

Dhaene, J. & Sundt, B. (1998). On approximating distributions by approximating their De Pril transforms. *Scandinavian Actuarial Journal*, 1-23.

Ekuma, O. (1998). *Aggregate claims distributions with emphasis on De Pril transforms.* Cand.scient. thesis. Department of Mathematics, University of Bergen.

Norberg, R. (1989). Experience rating in group life insurance. *Scandinavian Actuarial Journal*, 194–224.

Panjer, H.H. (1980). The aggregate claims distribution and stop-loss reinsurance. *Transactions of the Society of Actuaries* **32**, 523-535.

Panjer, H.H. (1981). Recursive evaluation of a family of compound distributions. *ASTIN Bulletin* **12**, 22-26.

Quenouille, M.H. (1949). A relation between the logarthmic, Poisson, and negative binomial series. *Biometrics* **5**, 162-164.

Sundt, B. (1992). On some extensions of Panjer's class of counting distributions. *ASTIN Bulletin* **22**, 61-80.

Sundt, B. (1993). *An introduction to non-life insurance mathematics.* (3. ed.) Verlag Versicherungswirtschaft e.V., Karlsruhe.

Sundt, B. (1995). On some properties of De Pril transforms of counting distributions. *ASTIN Bulletin* **25**, 19-31.

Sundt, B. & Jewell, W.S. (1981). Further results on recursive evaluation of compound distributions. *ASTIN Bulletin* **12**, 27-39.

Willmot, G.E. & Sundt, B. (1989). On evaluation of the Delaporte distribution and related distributions. *Scandinavian Actuarial Journal*, 101-113.

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