

MODELLING MORTGAGE INSURANCE AS A MULTI-STATE PROCESS

Greg Taylor

Taylor Fry Consulting Actuaries
Level 8, 30 Clarence Street
Sydney NSW 2000
Australia

Peter Mulquiney

Taylor Fry Consulting Actuaries
Level 8, 30 Clarence Street
Sydney NSW 2000
Australia

Professorial Associate,
Centre for Actuarial Studies
Faculty of Economics and Commerce
University of Melbourne
Parkville VIC 3052
Australia

Adjunct Professor in Actuarial Studies
Faculty of Commerce and Economics
University of New South Wales
Kensington NSW 2033
Australia

Phone: 61 2 9249 2901

Fax: 61 2 9249 2999

greg.taylor@taylorfry.com.au

Phone: 61 2 9249 2909

Fax: 61 2 9249 2999

peter.mulquiney@taylorfry.com.au

February 2006

Summary

Mortgage insurance claims are considered in Section 2 as an absorbing state in a Markov chain that involves transitions between the states Healthy, In arrears, Property in Possession, Property sold, Loan discharged, Claim. Section 3 considers the representation of this process by a cascade of five frequency GLMs, and a further GLM for claim size. These models are applied to the forecast of technical liabilities in Section 4, and the estimation of the associated forecast error in Section 5.

Keywords: Earned premium, forecast, forecast error, generalised linear model, loss reserve, Markovian, mortgage insurance, technical liabilities.

1. Introduction

This paper is concerned with modelling mortgage insurance claims experience. Most of its content applies to mortgage insurance generally, though in some places comment will be slanted towards insurance of residential, as opposed to commercial, mortgages.

Mortgage insurance indemnifies the lender against loss in the event of default under a mortgage leading to sale of the collateral property. A loss would occur if the proceeds of the sale, less its costs, were insufficient to defray the amount outstanding in respect of the mortgage loan. In the parlance of the financial markets, a mortgage insurance contract is a **credit default swap** (Saunders and Allen, 2002).

This line of business has a number of relatively unique properties:

- Policies provide multi-year coverage in return for a single premium.
- The claims experience is strongly influenced by variables that relate specifically to the housing sector of the economy.
- A claim occurs at the end of a defined sequence of events, somewhat different from those of other lines.

In view of the first of these, the premium in respect of any one policy needs to be earned over a number of years. Typically, an accounting standard will stipulate that it be earned in proportion with the “incidence of risk”, or some such phrase that broadly translates as “expected claim cost incurred”.

In practice, the **earning pattern** consists of a set of fixed percentages (e.g. 5% in the first year of loan, 15% in the second, and so on). An examination of claims history reveals, however, that earning patterns change from time to time, as conditions influencing the claims environment change. There is scope, therefore, to improve the accuracy of the assumed earning pattern by improved modelling of the claims experience.

Taylor (1994) introduced the application of a Generalised Linear Model (GLM) to mortgage insurance claims experience, using predictors that were external to the conventional triangles of claims experience. Ley and O’Dowd (2000)

continued this theme, but with extensions to allow for new products and other changes in the mortgage insurance market.

Driussi and Isaacs (2000) were less concerned with modelling of claims experience, but provide useful commentary on the market, and valuable data. Kelly and Smith (2005) again continue the framework established by Taylor, but introduce a detailed stochastic model of some of the external predictors.

It is characteristic of these papers that, wherever they introduce a model of claims experience, it is a single model, with number or amount of claims as the response variable. As noted above, each claim occurs at the end of a defined sequence of events (see Section 2), and it is possible to improve the modelling by extending it to recognise each component of the sequence.

The purpose of the present paper is to explore this extension of the modelling.

2. Claim process

2.1 Multi-state process

A mortgage loan will require the borrower to meet certain re-payment obligations. As long as these are met the loan will be designated **healthy**. If the borrower fails to do so at any time, the loan will be said to be **in arrears**.

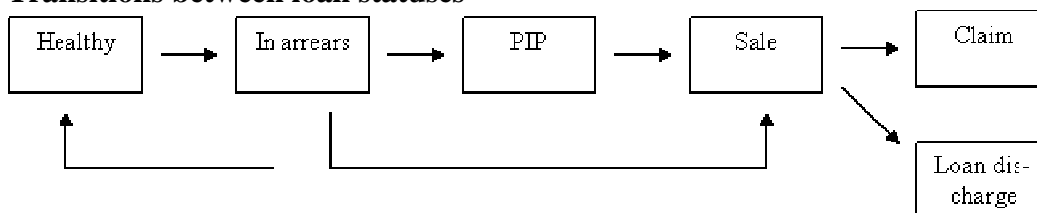
If this occurs the lender will usually take action aimed at recovering the arrears and restoring the loan to a healthy state. If, after a reasonable period, this has not been achieved, and appears unlikely to be, the borrower will take possession of the mortgaged property with a view to sale. The status of the loan will then be designated **PIP (property in possession)**.

Once this occurs the lender will almost certainly proceed to sale. If the proceeds of the sale, less the associated expenses (advertisement, cost of auction, etc) exceed the outstanding loan balance, the loan will be discharged. There will be no mortgage insurance claim. If, however, the loan balance exceeds the net proceeds of the sale, a claim for the balance will arise.

The transitions between loan statuses leading to a claim are therefore as represented in Figure 2.1. In fact, the figure includes two additional transitions, not covered by the above description. These reflect the facts that:

- Many cases of arrears are **cured** (returned to healthy status);
- Some borrowers in arrears voluntarily undertake sale, and thus by-pass the PIP status.

Figure 2.1
Transitions between loan statuses



The transition matrix associated with this process has the appearance set out in Figure 2.2.

Figure 2.2

Transition matrix for mortgage insurance claim process

	Healthy	In arrears	PIP	Sold	Loan discharged	Claim
Healthy	1-a	a				
In arrears	c	1-c-p-b	p	b		
PIP			1-s	s		
Sold					d	1-d
Loan discharged					1	
Claim						1

Note that “Loan discharged” and “Claim” are absorbing states, and the set of states {PIP, Sold, Loan discharged, Claim} is absorbing.

2.2 Date of claim occurrence

The usual definition of the date of occurrence of a claim is the date of the event generating the claim. In the case of mortgage insurance, there is some ambiguity as to this date.

Past papers have tended to treat it as the date on which sale of the property occurred and proceeds were found to be insufficient, i.e. that date on which the existence of the claim became known with certainty. However, an alternative school of thought might maintain that the event generating the claim was the policy’s transition to the arrears status.

The choice of date of occurrence may affect the evaluation of an insurer’s technical liabilities. These consist of the outstanding claims liability and the unexpired risk liability. These may in turn be expressed as:

- The liability for **incurred but not paid (IBNP)** (outstanding) claims; and
- The liability for **written but not incurred (WBNI)** claims (unexpired risk).

A change to the definition of date of occurrence causes a change in the dissection of total technical liabilities into these two components. It does not affect the expected value of the total liability, but may affect the provision made for it in the insurer’s accounts.

This is because the provision made for unexpired risk is often an **unearned premium provision**. If this is defined by reference to incidence of risk, as mentioned in Section 1, it will take the value

$$P \{1 - E[C_t]/E[C_n]\} \quad (2.1)$$

for a specific contract after t years of its n -year term have elapsed, where P is the premium payable under the contract net of acquisition and reinsurance expense, and C_t denotes the amount of claims incurred in the t years.

Let this quantity be denoted U_t . Then

$$U_t = P/E[C_n] \{E[C_n] - E[C_t]\} = (1+\pi) \{E[C_n] - E[C_t]\} \quad (2.2)$$

where the term within braces is the expected WBNI claims, and π is the profit margin contained in P expressed as a proportion of risk premium.

Thus unearned premium reserves include not only the relevant expected claim cost but also the associated profit margin. If this margin is positive, the unearned premium will exceed the expected claim cost. It follows that the later the choice of date of claim occurrence in Figure 2.1, the greater will be the estimated total amount of technical liabilities.

3. Model structure

3.1 Cascaded model

Figure 2.2 suggests a model consisting of a cascade of sub-models, each sub-model representing a single transition. The resulting models are as follows:

Sub-model 1: Probability of transition from Healthy to In arrears.

Sub-model 2: Probability of cure of arrears.

Sub-model 3: Probability of transition from arrears to PIP.

Sub-model 4: Probability of transition from PIP to Sold.

Sub-model 5: Probability of transition from Sold to Claim.

Sub-model 6: Distribution of claim sizes.

This is a slightly abbreviated version of the model suggested by Figure 2.2, as it omits the model of transition from In arrears to Sold. This case has, in fact been recognised by the insertion of an artificial transition to PIP infinitesimally prior to the sale. More accurately, therefore, Models 3 and 4 are as follows:

Sub-model 3: Probability of transition from arrears to **PIP or Sold**.

Sub-model 4: Probability of transition from PIP to Sold, with infinitesimal duration between the two indicating a direct transition from In arrears to Sold.

It is assumed that the cascaded model is **Markovian**.

3.2 Sub-model structures

Sub-models 1 to 5

Each of Sub-models 1 to 5 deals with a binomial process observed over calendar intervals $j=1,2,\dots,J$. These intervals are of equal but arbitrary length. We have found quarterly intervals useful. The binomial process is:

$$Y_{ij}^{(m)} \sim \text{Bin} [1, 1 - \exp \{u_{ij}^{(m)} \log [1 - p_{ij}^{(m)}]\}] \quad (3.1)$$

where $Y_{ij}^{(m)}$ denotes the binomial response of the i -th policy in Sub-model m ($=1,\dots,5$) over the j -th interval, $p_{ij}^{(m)}$ the associated probability of **transition**, and $u_{ij}^{(m)}$ the amount of time the policy is at risk of transition.

The quantities $u^{(m)}_{ij}$ take values in the range $[0,1]$. Values less than unity recognise incomplete intervals of exposure. For example, a loan that commences in an interval will be at risk of transition to arrears only for the remainder of that interval; likewise, a policy that terminates will be at risk of transition only for a fraction of the interval.

In general, if the interval j denotes $[j,j+1]$ and $p^{(m)}_{ij}$ relates to transition from status s_1 to s_2 , then

$$u^{(m)}_{ij} = t_2 - t_1 \quad (3.2)$$

where

$$t_1 = \max(j, \text{date first in status } s_1 \text{ in interval } j) \quad (3.3)$$

$$\begin{aligned} t_2 &= j+1, \text{ if transition to } s_2 \text{ occurs during the interval} \\ &= \min(j+1, \text{date of termination of the policy}), \text{ otherwise.} \end{aligned} \quad (3.4)$$

The above notation implicitly assumes that only one transition $s_1 \rightarrow s_2$ can occur in a single interval. In fact, multiple transitions may occur in the case of statuses Healthy and In arrears. To accommodate this within the notation is unnecessarily cumbersome. The notation is therefore left as above, and its extension to multiple transitions within an interval taken as obvious.

A transition probability defined in this way is the probability that the transition in question would occur over a complete quarter if no other transition intensities were operating. In actuarial terminology, it is an **independent** transition probability (Benjamin and Haycocks, 1970).

The probabilities in (3.1) may be naturally represented as:

$$\text{logit } p^{(m)}_{ij} = \sum_k \beta^{(m)}_k x_{ijk} \quad (3.5)$$

where x_{ijk} denotes value for observation (i,j) of the k -th predictor used in the 6 models (any one predictor may be used in more than one model, but not necessarily in all), and $\beta^{(m)}_k$ its associated coefficient.

Each model consisting of (3.1) and (3.5) is a **generalised linear model (GLM)** (McCullagh and Nelder, 1989).

Sub-model 6

The response variable $Y^{(6)}_{ij}$ is continuous, but may also be modelled with a GLM:

$$Y^{(6)}_{ij} \sim \text{EDF}(\mu_{ij}, q) \quad (3.6)$$

$$\log \mu_{ij} = \sum_k \beta^{(6)}_k x_{ijk} \quad (3.7)$$

where $\text{EDF}(\mu_{ij}, q)$ denotes a member of the **exponential dispersion family** with

$$E[Y^{(6)}_{ij}] = \mu_{ij} \quad (3.8)$$

$$\text{Var}[Y^{(6)}_{ij}] = (\phi/w_{ij}) \mu_{ij}^q \quad (3.9)$$

for constants ϕ, w_{ij} .

For any given i , there can, of course, be only one observation $Y^{(6)}_{ij}$, i.e. for a unique j . The index j has been retained in the model because of the possible dependency of the predictors x_{ijk} on it. Further comment on this appears in Section 3.3.

3.3 Predictors

The predictors x_{ijk} fall into three categories, which may be briefly designated:

- Policy variables, with sub-categories:
 - Static;
 - Dynamic;
- External variables;
- Manufactured risk indicators.

A reasonably comprehensive list of predictors appears in Section 4.1.

Policy variables assume values that are specific to individual policies, e.g. date of policy issue, loan-to-valuation ratio (LVR), etc. **Static** variables are those whose values remain constant over time (the two examples just given), while **dynamic** variables change over time (e.g. for a policy in arrears, number of quarters since transition to that status).

Some dynamic variables, such as this last one, will be simple mappings of calendar time (transition quarter). Hence the retention of the index j in the notation $Y^{(6)}_{ij}$, on which comment was made at the end of Section 3.2.

External variables are external to individual policies, and assume values that apply globally over all policies. They are indicators of the external economy, such as housing prices, interest rates, etc.

Manufactured risk indicators are specifically constructed from other variables as indicators of the risks inherent in specific sub-models. Examples of these are **Updated debt servicing ratio (DSR)** and **Potential claim size**, defined as follows:

$$\text{Updated DSR} = \frac{\text{Current outstanding principal} \times \text{interest rate}}{\text{Borrower's annual income} \times \text{average earnings growth factor}} \quad (3.10)$$

$$\begin{aligned} \text{Potential claim size} = & \text{Amount of arrears} \\ & \text{Less} \\ & \text{Principal repaid} \\ & \text{Less} \\ & [\text{Housing price growth factor} \times (1-q) - 1] \\ & \times \\ & \text{Original loan amount} / \text{LVR} \end{aligned} \quad (3.11)$$

where

- the borrower's annual gross income is as at issue of the loan (which the latest that will usually be on file);

- the average earnings and housing price growth factors are external variables, and relate to the period from inception of the loan to the current date;
- q is the proportion of property value lost in deadweight costs on sale.

If original DSR is defined as the ratio of the borrower's original per-period interest commitment to annual income, an obvious indicator of the likelihood of arrears, then updated DSR is a revised version allowing for any reduction of principal, change in interest rate, and change in income over the course of the loan to date.

Potential claim size comprises three members, of which the last two provide an estimate of the borrower's current equity in the property less the costs of realising it. The total quantity (3.11) is therefore an estimate of the amount of loss that would arise in the event of a claim.

4. Forecasting expected claim cost

4.1 Model fitting

Since the six models described in Section 3.2 are GLMs, they may be fitted to the data by means of GLM software. This may be computationally intensive. For example, in the case of an insurer with about 1¼ million policies in force, Model 1 involved about **23 million observations** (i.e. (i,j) combinations), one for each quarter that each policy was in force over a 9-year experience period (including policies no longer in force) and in the Healthy status (i.e. at risk of transition to arrears).

Otherwise, the application of the software is routine, with one exception. The appearance of the term $u^{(m)}_{ij}$ in (3.1) creates some awkwardness. However, substitution of (3.5) into the probability in (3.1), followed by a logit transform, yields (with suffixes temporarily suppressed for brevity)

$$\text{logit} (1 - \exp u \log (1-p)) = \text{logit} [1 - (1 + e^\eta)^{-u}] \quad (4.1)$$

where η denotes the linear predictor of the model, i.e. the right side of (3.5).

A little manipulation produces

$$\begin{aligned} \text{logit} (1 - \exp u \log (1-p)) &= \log [(1 + e^\eta)^u - 1] \\ &= [\log u + \eta] [1 + \frac{1}{2}(u-1) e^\eta + (1/6)(u-1)(u-2) e^{2\eta} + \dots] \end{aligned} \quad (4.2)$$

This shows that the logit transform of the probability in (3.1) may be represented by the (3.5) with a correction term of $\log u$, subject to a relative error of $O(\frac{1}{2}(u-1) e^\eta) = O(\frac{1}{2}(u-1) p/(1-p))$. The error is thus small for $p \rightarrow 0$ or $u \rightarrow 1$. In GLM terminology, the correction term amounts to adding an **offset** of $\log u$ into the model.

For Model 6, the value $q=1.5$ was found to be satisfactory in (3.9). This means that the distribution of claim size is right skewed but shorter-tailed than gamma.

For reasons of confidentiality (not to mention space), full numerical results are not presented here. However, a couple of isolated examples will be useful.

Example 4.1

Model of claim size

One of the strong predictors of average claim size is the potential claim size defined in (3.11). It contributes to the linear predictor (3.7) as follows:

$$0.71 X \log [\text{Potential claim size}/1000 + 5].$$

When account is taken of the model's log link, this means that forecast claim size is proportional to $[\text{Potential claim size}/1000 + 5]^{0.71}$, and so, for example, PIP cases with potential claim sizes of \$20,000 and \$5,000 respectively, but otherwise identical, will have forecast claim sizes in the ratio 1.9:1.

Example 4.2

Model of transition from Healthy to In arrears

The probability of transition from Healthy to In arrears status is found to depend on duration since commencement of the loan, after allowance has been made for all other predictors. In fact, this probability decreases with increasing development quarter (defined as the difference between the calendar quarter of commencement of the loan and the quarter of transition). From development quarter 1 onward, the decrease is linear.

Table 4.1 displays the predictors estimated to have statistically significant regression coefficients β in the 6 models. Table 4.2 provides some interpretive comment on some of the predictors.

Table 4.1
Statistically significant predictors

Predictor	Statistically significant predictors				
	Model 1: Healthy → In arrears	Model 2: In arrears → Cured	Models 3 and 4: In arrears → PIP	Model 5: PIP → Claim occurrence	Model 6: Claim size
Company within Group	*	*	*	*	*
State of Australia	*	*	*	*	*
Issue quarter		*	*		
Fixed/variable interest			*		
Geographic Classification Code	*	*	*		*
Lender Type	*	*	*		
Loan amount	*		*		
Loan Purpose	*		*	*	
Loan Type			*		
LVR	*	*	*	*	
Property type	*				*
Investment loan				*	*
Premium insured		*	*		
Principle and Interest			*		
Gross income			*		
NSR		*			
Economic environment risk coefficient	*		*	*	
Credit risk coefficient	*				
Full risk coefficient			*		
Reinsurer		*	*		
Duration since issue	*				
Duration since arrears		*	*		
Duration since PIP				*	*
Transition quarter	*	*	*	*	
Wage growth ^(a)			*		
10-year bond rate ^(b)	*		*		
GDP growth ^(a)	*				
House price index growth ^(a)				*	
Stock price index growth ^(a)	*		*		
Unemployment rate ^(b)		*	*	*	
Vacancy rate ^(b)		*			
Claim size potential ^(c)				*	*
Updated DSR ^(c)		*			

Notes: (a) Growth, as a percentage, measured from commencement of the loan to the quarter of transition.

(b) In the quarter of transition.

(c) Defined in Section 3.3.

Table 4.2
Interpretive comment on regression coefficients

Predictor	Comment
Company within Group	The same model covers (with variations) all companies within the Group
Issue quarter	Issue of the policy, i.e. commencement of the loan
Fixed/variable interest	Interest rate carried by the loan
Geographic Classification Code	Metropolitan, Rural, etc
Lender Type	Insurance company, finance company, etc
Loan Purpose	New residence construction, investment property, etc
Loan Type	
Property type	Free standing house, semi-detached, apartment, etc
Investment loan	
Premium insured	
Principle and Interest	
Gross income	
NSR	Net servicing ratio – the net income equivalent of DSR
Economic environment risk coefficient	A quantity developed by the company, measuring risk due to market conditions (i.e. not loan specific) at loan commencement
Credit risk coefficient	Lower coefficient \equiv greater risk
Full risk coefficient	A further quantity developed by the company, measuring total risk, including economic environment and loan specific risk, but excluding credit risk, at loan commencement
Reinsurer	Identity of quota share reinsurer
Duration since arrears	Since last transition to In arrears status
Duration since PIP	Since last transition to PIP status
Transition quarter	Calendar quarter of transition to new status (i.e. subject status of model)

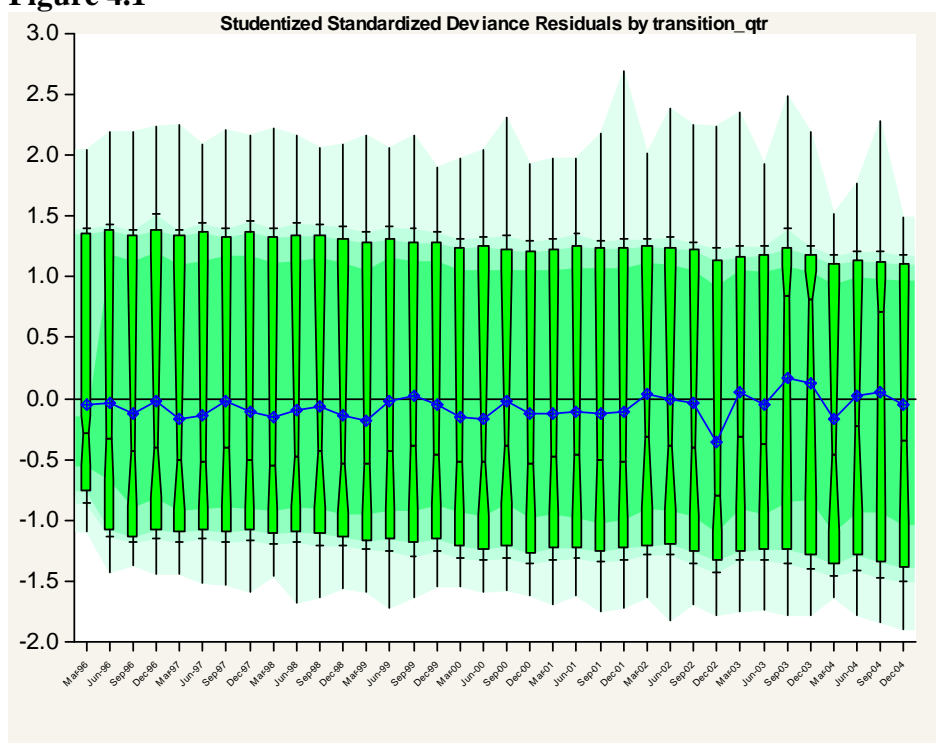
Many of the issues involved in fitting the model of claims experience are the same as for GLM loss reserving in any other line of business (see e.g. Taylor and McGuire, 2004).

For example, in each of the sub-models of transition from a source status to a target status, one has the choice of the following time-based predictors:

- Issue quarter;
- Duration since transition into the source status;
- Transition quarter.

These three time dimensions correspond to accident quarter, development quarter and calendar quarter in a more conventional loss reserving. As usual, collinearity is likely to prevent efficient inclusion of all three in a model, even if all three have some predictive power. In this case, it will be desirable to apply a little effort to determining which two have the greatest predictive power.

It is also desirable to examine the usual diagnostics of a GLM regression, such as residual plots, and triangles of observed to fitted value ratios by “accident period” and “development period”. Figure 4.1 provides an example of the former in respect of Sub-model 2, the probability of cure of arrears. It indicates centredness and homoscedasticity of the residuals with respect to transition quarter.

Figure 4.1

Forecasts of the external variables will be required, and it will usually be desirable that these be obtained from a separate econometric model, which might include housing prices, stock prices, gross domestic product, interest rates, inflation rates, unemployment rates, and possibly other variables.

4.2 Forecast claims experience

The conventional form of forecast in a loss reserve model consists of plugging future values of the predictors, such as accident period and development period, into the calibrated model, and taking the results as forecasts of future claim cash flows.

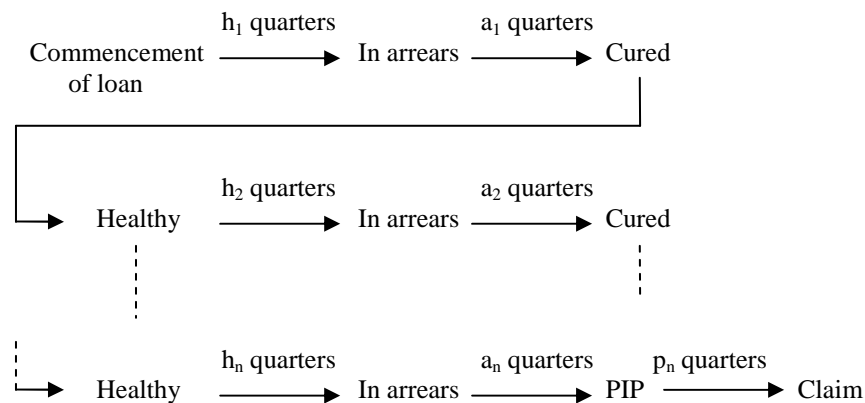
There are two reasons why this is neither feasible nor reliable on the present occasion. These are, in short, that:

- Such treatment of a cascaded model is unlikely to be computationally feasible;
- For specific reasons related to mortgage insurance, it is likely to produce a biased forecast.

Cascaded models

Note from Table 4.1 that Sub-models 1, 2 and 3 all include “development quarter” (duration since issue, duration since arrears, and duration since arrears respectively) as a predictor. Figure 4.2 illustrates the typical evolution of a claim.

Figure 4.2
Evolution of a claim



The probability of this occurrence is equal to the compound of $(h_1+a_1) + \dots + (h_n+a_n) + p_n$ single quarter probabilities, where $h_1, a_1, \dots, h_n, a_n, p_n$, and even n itself, are random variables. There are thus many combinations of single quarter probabilities, too many for feasible computation.

It is necessary therefore to simulate the experience of each policy in force at a valuation date. The simulated liability in respect of each policy will then be a random variable instead of the expected liability but, by the law of large numbers, the realization of the liability for a large number of policies will approximate its expected value.

The simulation will yield the forecast information set out in Table 4.3.

Table 4.3
Simulated forecast information

Policy no.	Status at valuation date	Valuation date + 1 quarter		Valuation date + 2 quarters		...
		Status at end of quarter	Claim payments in quarter	Status at end of quarter	Claim payments in quarter	
1						
2						
:						
:						

Multi-step transitions

The simulation of claims experience is not without complexities because, as was pointed out in Section 3.2, multi-step transitions can occur within a single quarter. Table 4.3 indicates that statuses are recorded only at end of quarter, but the transition between end-quarter statuses for consecutive quarters may be multi-step.

For example the status may Healthy at the ends of both quarters q and $q+1$, but there may have been any number of Healthy \rightarrow In arrears \rightarrow Healthy cycles within the quarter.

Treatment of this matter requires a continuous-time formulation of the discrete Markov process described by Figure 2.2, as abbreviated in Section 3.1. The discrete-time transition matrix for the abbreviated process is displayed in Figure 4.3, and its continuous-time equivalent in Figure 4.4.

In these matrices, the statuses Healthy, In arrears, PIP, Sold are denoted by h, a, p and s respectively. The status Claim is omitted, because the transition into it from Sold is not time dependent, but is a simple probability.

The quantity λ_{rs} ($s \neq t$) denotes the intensity of status transition $r \rightarrow s$ between the end of quarter t and the end of quarter t+1 (assumed constant over this interval) and, since p_{rs} is an independent transition probability (see Section 3.2),

$$\lambda_{rs} = -\log(1-p_{rs}) \quad (4.3)$$

Figure 4.3
Discrete-time transition matrix

	h	a	p	s
h	p_{hh}	p_{ha}	0	0
a	p_{ah}	p_{aa}	p_{ap}	0
p	0	0	p_{pp}	p_{ps}
s	0	0	0	1

Figure 4.4
Continuous-time transition matrix

	h	a	p	s
h	$-\lambda_{ha}$	λ_{ha}	0	0
a	λ_{ah}	$-\lambda_{ah}-\lambda_{ap}$	λ_{ap}	0
p	0	0	$-\lambda_{ps}$	λ_{ps}
s	0	0	0	0

Table 4.3 requires the quantities

$$P_{rs} = \text{Prob} [\text{status } s \text{ at end of quarter } t+1 \mid \text{status } r \text{ at end of quarter } t] \quad (4.4)$$

It is evident that the “whole quarter” transition matrix takes the form set out in Figure 4.5.

Figure 4.5
Whole quarter transition matrix

	h	a	p	s
h	P_{hh}	P_{ha}	P_{hp}	P_{hs}
a	P_{ah}	P_{aa}	P_{ap}	P_{as}
p	0	0	P_{pp}	P_{ps}
s	0	0	0	1

Consider the interval of time between the end of quarter t and the end of quarter t+1 and let the time elapsed since t be denoted by α where $0 \leq \alpha \leq 1$. Define

$$P_{rs}(\alpha) = \text{Prob} [\text{status } s \text{ at } \alpha \mid \text{status } r \text{ at end of quarter } t] \quad (4.5)$$

Then $P_{rs}(1) = P_{rs}$ as defined in (4.4).

Values of $P_{rs}(1)$ were calculated by solving the Kolmogorov Forward Equation

$$\partial W(\alpha)/\partial \alpha |_{\alpha=1} = W(\alpha) Q \quad (4.6)$$

subject to initial conditions $W(0)$, where $W(\alpha)$ is the partial quarter transition matrix and Q is the continuous-time transition matrix (Figure 4.4). The Kolmogorov Forward Equation was solved using *Mathematica* and the values of $P_{rs}(1)$ are set out in Appendix A. These are the probabilities used to simulate change in status from one quarter-end to the next.

If policy m attains end-quarter status s for the first time in quarter j^* , the further transition to either Claim or Discharged is simulated. If the latter, simulation for the policy is terminated; if Claim, a claim size Y is simulated from Sub-model 6, as described in Appendix B, and then simulation for the policy is terminated. In the case of discharge, Y is set to zero.

Let $L_i^{(j)}$ denote claim payments for policy i in quarter j . Then the simulated value is

$$\begin{aligned} L_i^{(j)} &= Y, \text{ for } j=j^* \\ &= 0, \text{ otherwise} \end{aligned} \quad (4.7)$$

The total liability for future claims is then

$$L = \sum_{i,j} L_i^{(j)} \quad (4.8)$$

where the summation runs over all values of i and future values of j . The total liability can be dissected in any way required (e.g. according to year of policy issue) simply by summing over subsets of (i,j) in (4.8).

Stochastic future parameters

There is a second reason why simulation may be a desirable form of loss reserving. Let

$$L_{\cdot i} = \sum_j L_i^{(j)} = \text{simulated liability in respect of policy } i \quad (4.9)$$

$$L_{j \cdot} = \sum_i L_i^{(j)} = \text{simulated liability cash flow in respect of calendar quarter } j \quad (4.10)$$

so that

$$L = \sum_i L_{\cdot i} = \sum_j L_{j \cdot} \quad (4.11)$$

The forecast of $L_{j \cdot}$ will depend on the estimates $\hat{\beta}_k^{(m)}$ of the model parameters $\beta_k^{(m)}$, $m=1, \dots, 6$, and also on the x_{ijk}^* taken in future quarter j by the predictors x_{ijk} . With these dependencies made explicit, (4.8) suggests the forecast

$$L^* = \sum_{i,j} L_i^{(j)}(\hat{\beta}, x_{ij}^*) \quad (4.12)$$

where $\hat{\beta}$ is the vector of values $\hat{\beta}_k^{(m)}$ over all m , and x_{ij}^* the vector with components x_{ijk}^* ,

In the terminology of Section 3.3, x^*_{ij} may be decomposed into sub-vectors:

$$x^*_{ij}{}^T = [\xi^*_{i}{}^T, \zeta^*_{ij}{}^T, z^*_{ij}{}^T] \quad (4.13)$$

where ξ^*_{i} contains static policy variables (which do not depend on j), $\zeta^*_{ij}{}^T$ contains dynamic policy variables, $z^*_{ij}{}^T$ contains forecasts of external variables and manufactured risk variables, and T denotes the operation of transposition.

Dynamic policy variables depend on j but are simple mappings of it (e.g. number of quarters since transition to In arrears status), whereas external variables and manufactured risk variables require forecast. Examples of such predictors, from Table 4.1, are house price growth, stock price growth, and generally any predictor taken from an economic time series.

The z^*_{ijk} are forecasts of quantities z_{ijk} that may be viewed here as themselves random variables. In this case an unbiased forecast of L will be

$$L^* = \sum_{i,j} E[L^{(i)}_i(\hat{\beta}, x_{ij})] = \sum_{i,j} E[L^{(i)}_i(\hat{\beta}, \xi^*_{i}, \zeta^*_{ij}, z_{ij})] \quad (4.14)$$

Note that, in general, this is not equal to the alternative forecast

$$L'^* = \sum_{i,j} L^{(i)}_i(\hat{\beta}, E[x_{ij}]) = \sum_{i,j} L^{(i)}_i(\hat{\beta}, \xi^*_{i}, \zeta^*_{ij}, E[z_{ij}]) \quad (4.14a)$$

Indeed, Jensen's inequality (see e.g. Royden, 1968) indicates that (4.14a) may be distinctly biased relative to (4.14).

Jensen's inequality. Let A be a real interval and let $f:A \rightarrow \mathcal{R}$ be convex in the sense that $\alpha f(x_1) + (1-\alpha)f(x_2) \geq f(\alpha x_1 + (1-\alpha)x_2)$ for all $x_1, x_2 \in A$ and all $0 < \alpha < 1$. Let X be a random variable defined on A . Then

$$E[f(X)] \geq f(E[X]) \quad (4.15)$$

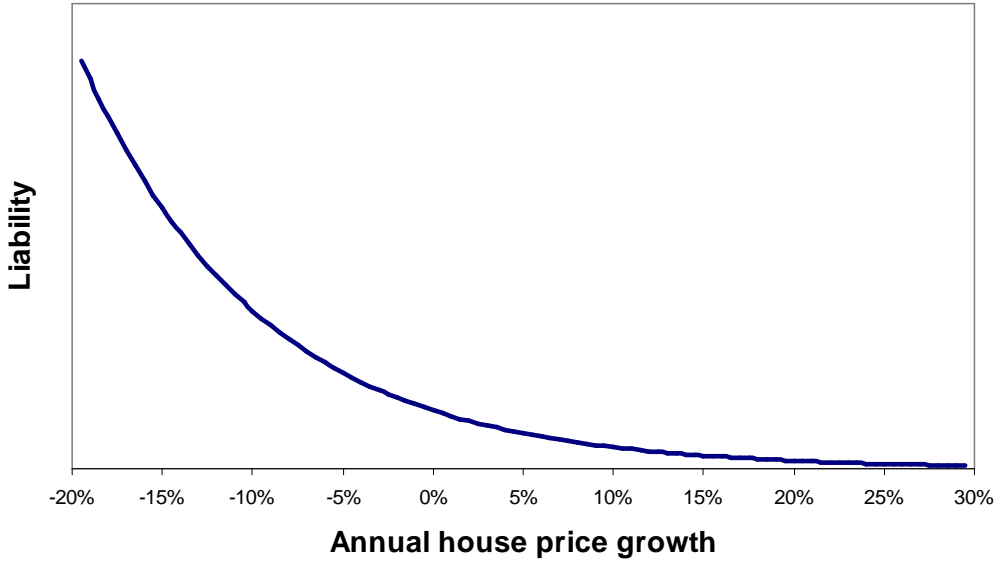
with equality if and only if either f is linear (i.e. equality in the definition of convexity) or the distribution of X is concentrated at a single point. \square

Note that a sufficient condition for f to be convex is that $f'' \geq 0$.

Now consider the functional dependency of L^* on z_{ij} , a sub-vector of economic time series variables, such as house price increases or stock price increases. The functional dependency on some of these quantities involves considerable convexity.

This is particularly the case for house price increases, whose influence on liability is illustrated by Figure 4.6. If z_{ijk} denotes future annual house price increase, then $\partial^2 L / \partial z_{ijk}^2 > 0$, and Jensen's inequality applies.

Figure 4.6
Dependence of liability on future house price increases



If other predictors are fixed at their expected values for the moment, then (4.15) yields

$$E[L(\hat{\beta}, x_{ij})] \geq L(\hat{\beta}, \xi^*_i, \zeta^*_{ij}, E[z_{ij}]) \quad (4.16)$$

What this means is that estimating liability by taking mean forecasts of future house price increases and plugging them into a liability formula as if certain will inevitably under-estimate. The extent of the under-estimation will increase with:

- the convexity of the liability with respect to future house price growth; and
- the dispersion of the distribution of these increases.

If the liability displays the same sort of convexity with respect to other predictors x_{ijk} , the failure of the “plug-in” approach to predictors of the future economic time series type will be even greater. This sort of result has recently been observed empirically by Kelly and Smith (2005).

This feature of loss reserving is virtually unique to mortgage insurance. The forecasting of liability in other lines of business does not usually depend heavily on future economic time series. The main exception to this is future claims inflation, but commonly the convexity of liability with respect to this variable is less than its mortgage insurance counterpart, and the dispersion in the future time series will often be less than in mortgage insurance.

The estimation of liability by means of simulation avoids this form of under-estimation. The future values of z_{ij} are simulated from a model of the external economic influences. It is likely that the components z_{ijk} will not all be independent. In any case, let $z_{ij}^{(r)}$, $r=1, \dots, R$, denote the r -th simulated replicate.

Then, by (4.14), an unbiased forecast of L is

$$L^* = R^{-1} \sum_{i,j,r} E[L^{(j)}_i(\hat{\beta}, \xi^*_i, \zeta^*_{ij}, z_{ij}^{(r)})] \quad (4.17)$$

The computation in (4.17) is substantial. It involves R simulations of the future experience of each policy i over future quarters j , with different replicates of external economic variables in each of the R simulations.

5. Forecast error

5.1 Fast bootstrap

Estimation of the forecast error contained in L^* for such a complex model as defined in Section 3 will not be feasible other than by some form of bootstrap. However, the conventional form of bootstrapping, with the re-fitting of six models to each pseudo-data set is not likely to be feasible.

A “fast bootstrap” has therefore been used, in which the steps of generating pseudo-data sets and re-fitting the model to them have been by-passed. Instead, pseudo-estimates of the model parameters are obtained simply by perturbing the original estimates, with due reference to their standard errors.

More precisely, the procedure is as follows. It is supposed that the vector of parameter estimates $\hat{\beta}$ is accompanied by an estimate V of the associated covariance matrix. The standard GLM software packages (SAS GENMOD, Emblem, etc) produce this.

If β denotes the vector of parameters estimated by $\hat{\beta}$ (let its dimension be q), suppose that

$$\hat{\beta} \sim N(\beta, V) \quad (5.1)$$

This is asymptotically true for large samples for the maximum likelihood estimate $\hat{\beta}$. It is not to assume, of course, that the observations $Y^{(m)}_{ij}$ are normally distributed.

Diagonalise V :

$$V = P^T D P \quad (5.2)$$

where D is diagonal and P is orthogonal. Then

$$P \hat{\beta} \sim N(P\beta, D) \quad (5.3)$$

Thus, the components of $P \hat{\beta}$ may be sampled independently. Let $\varepsilon^{(n)}$, $n=1, \dots, N$, be a q -vector of samplings from the unit normal distribution., and define

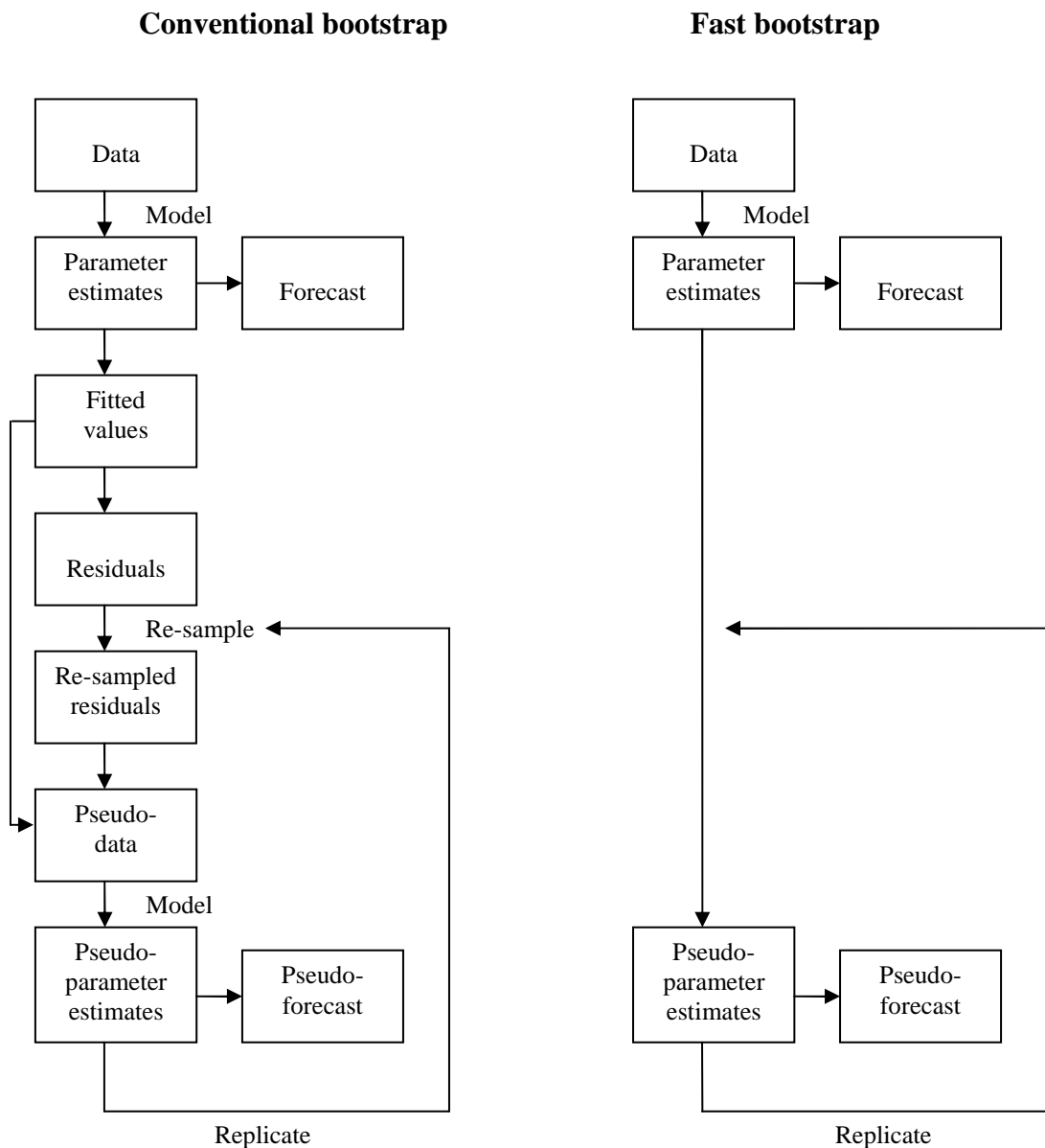
$$\hat{\beta}^{(n)} = P^T (P \hat{\beta} + D \varepsilon^{(n)}) = \hat{\beta} + P^T D \varepsilon^{(n)} \quad (5.4)$$

Then the $\hat{\beta}^{(n)}$ are replicates of $\hat{\beta}$ in the sense of satisfying (5.1). These generate replicates $L^{*(n)}$ of L^* whose dispersion reflects the parameter error in the

forecast L^* , i.e. the uncertainty in L^* induced by uncertainty in $\hat{\beta}$. Note that this is computationally demanding, requiring N replicates of the previous R simulations.

Figure 5.1 compares the fast bootstrap procedure with the conventional one, illustrating how the former omits the data re-sampling steps.

Figure 5.1
Fast and conventional bootstraps



5.2 Estimation of forecast error

Components of forecast error

Consider the forecast L^* in (4.17). Let L denote the true (yet to be observed) value of the liability. The forecast error is

$$\Delta = L - L^* \quad (5.5)$$

Noting that $L^* = E[L|\hat{\beta}]$, one may decompose this expression as follows:

$$\Delta = \{L - E[L]\} - \{E[L|\hat{\beta}] - E[L]\}$$

which may be decomposed further:

$$\begin{aligned} \Delta &= \{L - E[L]\} - \{E[L|\mathcal{M}, \hat{\beta}] - E[L|\mathcal{M}]\} - \{E[L|\mathcal{M}] - E[L]\} \\ &= \Delta_{\text{Proc}} - \Delta_{\text{Pa}} - \Delta_{\text{Sp}}, \text{ say} \end{aligned} \quad (5.6)$$

where $E[L|\hat{\beta}]$ has simply been re-written as $E[L|\mathcal{M}, \hat{\beta}]$, recognizing the model structure to which the parameter estimates $\hat{\beta}$ belong.

The three terms on the right side of (5.6) may be recognized as **process error**, **parameter error**, and **model specification error** in the standard terminology. In fact, it is useful to decompose the second of these even further:

$$\begin{aligned} \Delta_{\text{Pa}} &= E[L|\mathcal{M}, \hat{\beta}] - E[L|\mathcal{M}] \\ &= \{E[L|\mathcal{M}, \hat{\beta}] - E[L|\mathcal{M}, \hat{\beta}, \{x_{ij}\}]\} + \{E[L|\mathcal{M}, \hat{\beta}, \{x_{ij}\}] - E[L|\mathcal{M}]\} \\ &= \Delta_{\text{Pred}} + \Delta_{\text{Par}}, \text{ say} \end{aligned} \quad (5.7)$$

where $\{x_{ij}\}$ denotes the set of predictors over all i, j , and so (5.7) decomposes parameter error into that due to variation in the parameters β , and that due to variation in the predictors.

Recall from Section 4.2 that that x_{ij} consists of ξ_i , ζ_{ij} and z_{ij} , of which the first two are non-random (other than captured in process error), so Δ_{Pred} and Δ_{Par} could equally well be defined with $\{z_{ij}\}$ written in place of $\{x_{ij}\}$.

By (5.6) and (5.7),

$$\Delta = \Delta_{\text{Proc}} - [\Delta_{\text{Pred}} + \Delta_{\text{Par}}] - \Delta_{\text{Sp}} \quad (5.8)$$

The four members on the right are stochastically independent, and so the **mean square error of prediction** is

$$\text{MSEP}[L] = \text{Var}[\Delta_{\text{Proc}}] + \text{Var}[\Delta_{\text{Pred}}] + \text{Var}[\Delta_{\text{Par}}] + \text{Var}[\Delta_{\text{Sp}}] \quad (5.9)$$

Estimation

As usual, estimation of model specification error is problematic. It falls outside the scope of the statistical modeling considered here and, while an issue of substance, is not discussed further in this paper.

The other three components of forecast error can be estimated from the simulations defined in Section 4.2. This will require multiple replicates of the forecast (4.17). Let the n -th of these be denoted by

$$L^{*(n)} = \mathbf{R}^{-1} \sum_{i,j,r} E[L]_i^{(j)}(\hat{\beta}^{(n)}, \xi_{*i}^*, \zeta_{*ij}^{*(n)}, z_{ij}^{*(r)}), n=1, \dots, N \quad (5.10)$$

where $\hat{\beta}^{(n)}$ is the n -th drawing of the vector of pseudo-parameters, and the notation $\zeta_{ij}^{*,(n)}$ recognizes that the dynamic predictors will be replicate-dependent.

Note also that the replicates $z_{ij}^{(r)}$ of the variable of external variables are **not** dependent on the parameter vector β , and so may be repeated in each pseudo-estimate $\hat{\beta}^{(n)}$.

It is convenient to denote the summand in (5.10) as $L^{*(n,r)}$, $n=1,\dots,N$, $r=1,\dots,R$. These $N \times R$ quantities form an empirical distribution of L . In general, the mean of the distribution \bar{L}^* will not equal L^* from (4.17).

If so desiring, one may adjust to a mean of L^* by re-scaling the empirical distribution, replacing $L^{*(n,r)}$ by $L^{*(n,r)} \times L^*/\bar{L}^*$. However, the difference between L^* and \bar{L}^* may arise from convexity of L as a function of $\hat{\beta}$, in the same way as discussed in Section 4.2, and it may be prudent to retain the unadjusted empirical distribution given by the $L^{*(n,r)}$.

In the following, it will be assumed that re-scaling is not carried out, but the analysis is easily extended to the case where it is. Then $\text{MSEP}[L] - \text{Var}[\Delta_{Sp}] = \text{Var}[\Delta_{Proc}] + \text{Var}[\Delta_{Pred}] + \text{Var}[\Delta_{Par}]$ is estimated by

$$s^2(\text{Proc}, \text{Pred}, \text{Par}) = (NR)^{-1} \sum_{n,r} [L^{*(n,r)} - \bar{L}^*]^2 \quad (5.11)$$

Let

$$\bar{L}^{*(n, \cdot)} = R^{-1} \sum_r L^{*(n,r)} \quad (5.12)$$

$$\bar{L}^{*(\cdot, r)} = N^{-1} \sum_n L^{*(n,r)} \quad (5.13)$$

Then (5.11) may be decomposed by analysis of variance:

$$s^2(\text{Proc}, \text{Pred}, \text{Par}) = s^2(\text{Proc}, \text{Par}) + s^2(\text{Pred}) \quad (5.14)$$

where

$$s^2(\text{Proc}, \text{Par}) = N^{-1} \sum_n [\bar{L}^{*(n, \cdot)} - \bar{L}^*]^2 \quad (5.15)$$

$$s^2(\text{Pred}) = N^{-1} \sum_n R^{-1} \sum_r [L^{*(n,r)} - \bar{L}^{*(n, \cdot)}]^2 \quad (5.16)$$

As the notation suggests, $s^2(\text{Proc}, \text{Par})$ estimates $\text{Var}[\Delta_{Proc}] + \text{Var}[\Delta_{Par}]$ and $s^2(\text{Pred})$ estimates $\text{Var}[\Delta_{Pred}]$.

The process error is automatically present in the original forecast L^* since this is derived by simulation. Hence $s^2(\text{Proc}, \text{Par})$ is not separable into its two components without a little effort.

However, an estimate of $\text{Var}[\Delta_{Proc}]$ may be obtained from (5.15) calculated with $\hat{\beta}^{(n)} = \hat{\beta}$ for each n . This eliminates parameter error, and the only remaining variation is that arising from the differing simulated experiences with different n .

6. Miscellaneous other matters

6.1 Earning of premium

Section 1 notes the typical requirement that premium be earned over the term of a policy in proportion with expected claim cost incurred.

For policies with inception quarter J , define the **earning pattern** over quarters $J, J+1, J+2, \dots$ to be the vector of expected claims incurred in those quarters expressed as a proportion of the total incurred for those policies. Denote the proportion associated with quarter $J+d$ by $\pi_d^{(J)}$, where d will be referred to as **development quarter**.

Estimation of the earning pattern will require a definition of when claims are incurred. In general, a claim is incurred at the time of occurrence of the event causing it. However, in mortgage insurance the definition of this event is less obvious than in other lines of business.

It may reasonably be taken as the PIP event (or sale in the case that this is voluntary), or the commencement the arrears leading to the claim. Other definitions may be possible.

For a policy i with inception quarter J , having completed D development quarters at a valuation date, an estimate future claim payments $L^{(J+d)}_i$, $d=D+1, D+2, \dots$, is given by (4.10). **Actual** claim payments will be available for past development quarters $d=0, 1, \dots, D$. For present purposes, let these be denoted by $L^{(J+d)}_i$ also.

This gives a complete vector of claim payments $\{L^{(J+d)}_i, d=0, 1, 2, \dots\}$ for policies from inception quarter J . If an earning pattern is to be estimated, it will be necessary to re-cast these payments according to their dates incurred rather than dates paid.

If, for example, date incurred is taken as PIP event, then all payments will need to be re-assigned from payment quarter to PIP quarter. For past payments, these dates will be factual. For forecast future payments, there will be simulated PIP and payment quarters, according to which the re-assignment may be made.

This re-assignment will convert the payment vector $\{L^{(J+d)}_i, d=0, 1, 2, \dots\}$ to a vector of incurred amounts $\{I^{(J+d)}_i, d=0, 1, 2, \dots\}$ where

$$\sum_d L^{(J+d)}_i = \sum_d I^{(J+d)}_i \quad (6.1)$$

The earning pattern may now be calculated as

$$\pi_d^{(J)} = I^{(J+d)}_i / \sum_d I^{(J+d)}_i, \quad d=0, 1, 2, \text{etc} \quad (6.2)$$

where

$$I^{(J+d)}_i = \sum_i I^{(J+d)}_i \quad (6.3)$$

the summation over i running over all policies with inception quarter J .

Note that the notation $\pi_d^{(J)}$ expresses dependency on J . This dependency is induced by changes in the mix of policies (i.e. changing distribution of policy variables) with changing inception quarter, and changes in the economy (i.e. changing external variables) over time.

Consider, for example, the case represented in Figure 6.1, in which the housing market proceeds steadily over quarters 1 to 8, then crashes over the following 4 quarters, then recovers strongly.

Figure 6.1
Housing price index

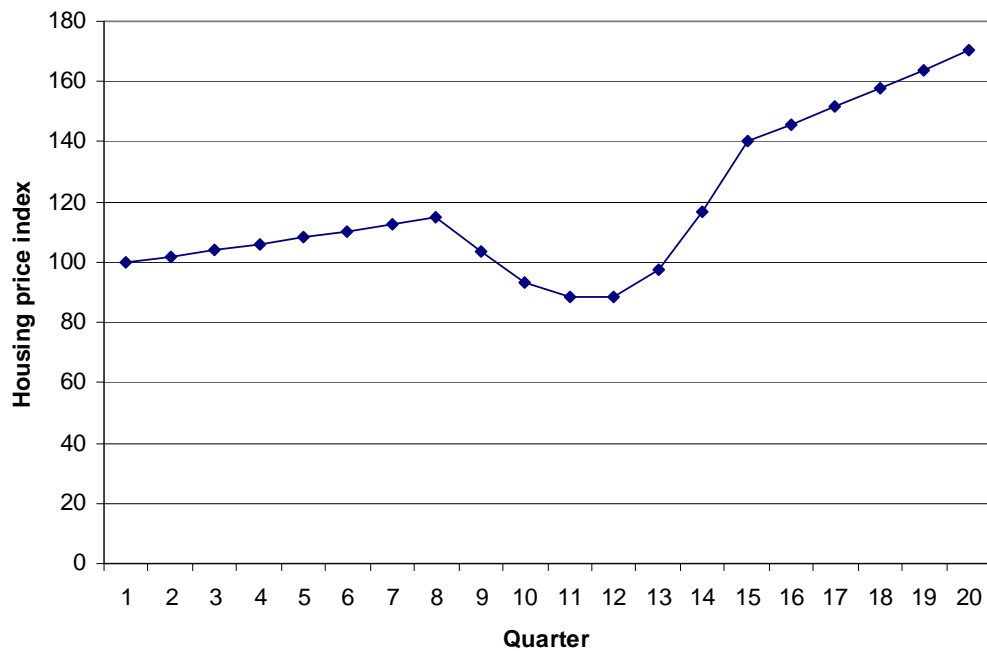


Figure 6.2 illustrates how claims might be incurred over time for a number of inception quarters, and Figure 6.3 converts these into earning patterns.

Figure 6.2
Expected incurred claims by inception quarter

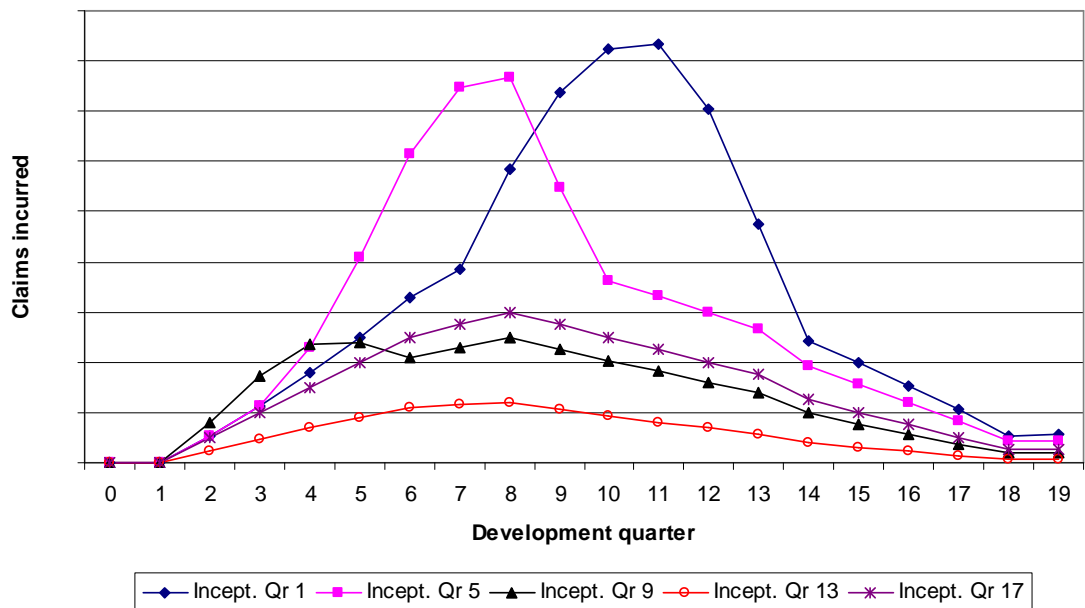
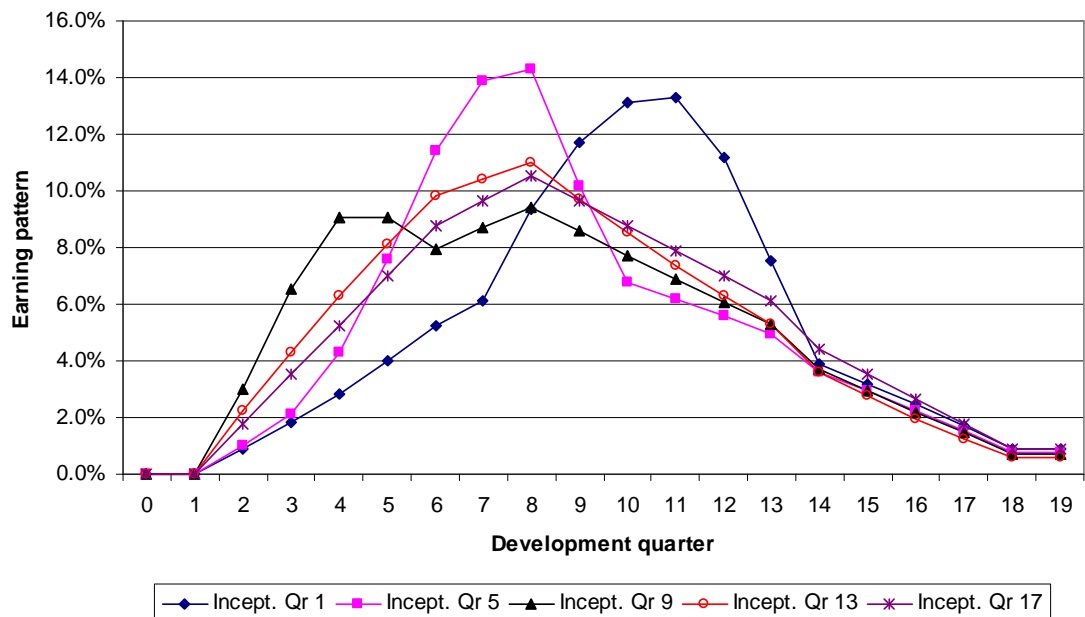


Figure 6.3
Earning pattern by inception quarter



Inception quarter 17 experiences a steady 4% per quarter growth in house prices, and may be taken as the standard in interpreting these plots.

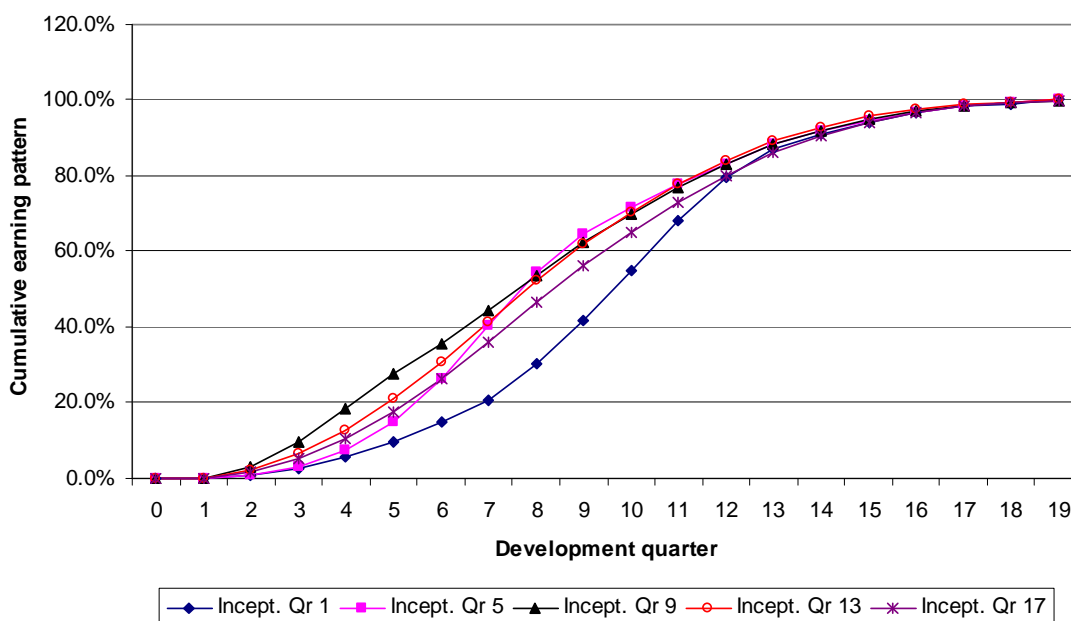
Note that inception quarter 13 experiences a much lower level of claims. This is because the recovery in housing prices, with high growth rates, almost coincides with that quarter. As a result, housing prices have increased by 50% after 4 development quarters, compared with 17% for inception quarter 17. The corresponding increases after 8 development quarters are 75% and 32%.

Inception quarters 1 and 5 have similar expected claims experiences except that the latter is displaced to the left. This is because the property price collapse affects it 4 development quarters earlier. Both of these inception quarters are associated with relatively heavy claims experience, also due to the collapse.

The earning patterns in Figure 6.3 simply reproduce Figure 6.2 but with each inception quarter re-scaled to total 100%. This leads to slightly paradoxical results, such as the fact that premium for inception quarter 13 is earned considerably more slowly over the early development quarters than for inception quarter 9 (see Figure 6.4), despite the fact that the claims experience of the former is much the lower, and therefore the profit margin much the higher.

Of course, although the earning pattern for inception quarter 13 releases lesser percentages of the profit margin than in the case of inception quarter 9, the quantum of profit margin is much greater in the former case if there has been no change in premium rates.

Figure 6.4
Cumulative earning pattern by inception quarter



6.2 Policy termination

It is common for mortgage insurance policies to be terminated before their term has elapsed. This occurs usually because the property is sold or the loan otherwise repaid before the term of the loan has expired.

It is necessary to allow for this form of exit from experience in calibrating the Models 1 to 5 of Section 3.1. Specifically, in (3.1) policy i must be assigned $u_{ij}^{(m)}$ = fraction of calendar quarter j up to policy termination, when termination occurs in that quarter.

This ensures correct estimation of transition probabilities. However, there is a further question as to whether or not a valuation of technical liabilities should

anticipate future terminations. In Australia at least, it is typical for terminations not to attract any premium refund after one year's policy duration. This means that under-estimation of termination rates (such as not anticipating any) will lead to a conservative valuation.

The action taken in this area will be a matter of taste and/or statute. In the event that policy terminations are to be anticipated, it will be necessary to estimate termination rates, and use these to simulate survival or otherwise over future periods of each policy in force at the valuation date.

Then the simulation of $L_i^{(j)}$ in (4.7) will be carried out only for those quarter j for which policy i is simulated to survive.

Appendix A

Evaluation of multi-step transition probabilities

The Kolmogorov Forward Equation (4.6) was solved using *Mathematica* to produce the following entries in Table 4.5:

$$P_{hh} = \frac{(r_2 - \lambda_{ha}) \exp -r_1 - (r_1 - \lambda_{ha}) \exp -r_2}{r_2 - r_1} \quad (A.1)$$

$$P_{ha} = \frac{\lambda_{ha} (\exp -r_1 - \exp -r_2)}{r_2 - r_1} \quad (A.2)$$

$$P_{hp} = \frac{-\lambda_{ha}\lambda_{ap}\{\exp -r_2[r_2 - r_1 - (\lambda_{ha} + \lambda_{ah} + \lambda_{ap} - 2\lambda_{ps})] + \exp -r_1[r_2 - r_1 - (\lambda_{ha} + \lambda_{ah} + \lambda_{ap} - 2\lambda_{ps})]\}}{2(r_2 - r_1)(\lambda_{ha}\lambda_{ap} - \lambda_{ha}\lambda_{ps} - \lambda_{ah}\lambda_{ps} - \lambda_{ap}\lambda_{ps} + \lambda_{ps}^2)} + \frac{-\lambda_{ha}\lambda_{ap}(-2\lambda_{ha}^2 - 4\lambda_{ha}\lambda_{ah} - 2\lambda_{ah}^2 + 4\lambda_{ha}\lambda_{ap} - 4\lambda_{ah}\lambda_{ap} - 2\lambda_{ap}^2)}{2[\exp \lambda_{ps}](r_2 - r_1)^2(\lambda_{ha}\lambda_{ap} - \lambda_{ha}\lambda_{ps} - \lambda_{ah}\lambda_{ps} - \lambda_{ap}\lambda_{ps} + \lambda_{ps}^2)} \quad (A.3)$$

$$P_{ah} = \frac{\lambda_{ah} (\exp -r_1 - \exp -r_2)}{r_2 - r_1} \quad (A.4)$$

$$P_{aa} = \frac{(r_2 - r_1 + \lambda_{ha} - \lambda_{ah} - \lambda_{ap}) \exp -r_1 + (r_2 - r_1 - \lambda_{ha} + \lambda_{ah} + \lambda_{ap}) \exp -r_2}{2(r_2 - r_1)} \quad (A.5)$$

$$P_{ap} = \lambda_{ap} (\exp \lambda_{ps}) [\exp -r_2 + \exp -r_1] \times \frac{[(\lambda_{ha} - \lambda_{ps})(r_2 - r_1)^2 + (r_2 - r_1) + (-\lambda_{ha}^2 - \lambda_{ha}\lambda_{ah} + \lambda_{ha}\lambda_{ap} + \lambda_{ha}\lambda_{ps} - \lambda_{ah}\lambda_{ps} - \lambda_{ap}\lambda_{ps})] - \lambda_{ap}[2\lambda_{ha}^3 + 4\lambda_{ha}^2\lambda_{ah} + 2\lambda_{ha}\lambda_{ah}^2 - 4\lambda_{ha}^2\lambda_{ap} + 4\lambda_{ha}\lambda_{ah}\lambda_{ap} + 2\lambda_{ha}\lambda_{ap}^2 - 2\lambda_{ha}\lambda_{ps}^2 - 4\lambda_{ha}\lambda_{ah}\lambda_{ps} - 2\lambda_{ah}\lambda_{ps}^2 + 4\lambda_{ha}\lambda_{ap}\lambda_{ps} - 4\lambda_{ah}\lambda_{ap}\lambda_{ps} - 2\lambda_{ap}\lambda_{ps}^2]}{2(r_2 - r_1)^2}$$

$$[2 (\exp \lambda_{ps}) (r_2 - r_1)^2 (-\lambda_{ha}\lambda_{ap} + \lambda_{ha}\lambda_{ps} + \lambda_{ah}\lambda_{ps} + \lambda_{ap}\lambda_{ps} - \lambda_{ps}^2)] \quad (A.6)$$

$$P_{pp} = \exp - \lambda_{ps} \quad (A.7)$$

where

$$r_1 = 1/2 \{ \lambda_{ha} + \lambda_{ah} + \lambda_{ap} + [(\lambda_{ha} + \lambda_{ah} + \lambda_{ap})^2 - 4 \lambda_{ha} \lambda_{ap}]^{1/2} \} \quad (A.8)$$

$$r_2 = 1/2 \{ \lambda_{ha} + \lambda_{ah} + \lambda_{ap} - [(\lambda_{ha} + \lambda_{ah} + \lambda_{ap})^2 - 4 \lambda_{ha} \lambda_{ap}]^{1/2} \} \quad (A.9)$$

Appendix B

Simulation from EDF quasi-likelihood

Let $Y \sim \text{EDF}(\mu, q)$, as defined in the GLM (3.6)-(3.9). Then, with the log link appearing in (3.7) taken into account, the log-likelihood of Y is

$$L(y, \mu) = \mu^{-q} \varphi^{-1} [\mu y / (1-q) - \mu^2 / (2-q)] \quad (B.1)$$

The **deviance residual** associated with observation Y is defined as:

$$d(Y) = \text{sgn}(Y - \hat{Y}) \{ \varphi [L(Y, \hat{Y}) - L(Y, Y)] \}^{1/2} \quad (B.2)$$

where \hat{Y} is the value fitted to Y by the GLM.

The **standardised deviance residual** is

$$d^S(Y) = d(Y) / \{ \sum d^2(Y) \}^{1/2} \quad (B.3)$$

where the summation runs over the deviance residuals for all observations Y in the data set.

The parameter φ is called the **scale parameter**, and is estimated by $\sum d^2(Y) / (n-p)$, where n is the number of observations modelled and p the number of parameters in the model.

It is shown by Pierce and Schafer (1986) that the standardised deviance residuals have a distribution that is close to unit normal. Therefore, a random drawing Y from (quasi-)log-likelihood (B.1) with $\mu = \hat{Y}$ may be obtained by taking a random drawing $d^S \sim N(0,1)$, and calculating Y as the solution of equation (B.2) for given φ and \hat{Y} .

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