Excessive Trading:
A Role of Local and Dynamic Learning*
(Incomplete: Comments are welcome)

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Abstract

We present a model of excessive trading based on dynamic and local learning. We analyze how information is aggregated through local interactions, and how such a learning process affects allocations over time. In the absence of private information about the value of the traded asset, the allocation monotonically approaches the efficient allocation over time. With dispersed private information, richer patterns of dynamic trading activity emerge due to small sample errors in local learning. In particular, trading can be excessive in that the allocation moves away from the efficient allocation. For the asset subject to a high level of information asymmetry, both U- and inverted-U patterns of trading activity are possible. Moreover, the fluctuation in trading activity is amplified when the expected frequency of trading goes up.

Keywords: Asymmetric information, Dynamic trading, Information aggregation.

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1 Introduction

Trading in financial markets is very active. Recent study shows that the turnover rate for stocks on the NYSE is well above 100% and the level of volume has increased significantly (Chordia et al 2011). Many, both in academia and practice, argue that trading activity appears excessive for financial assets including stocks, foreign exchanges, and derivatives. For example, while rich dynamic patterns of stock trading such as volume autocorrelation and U-shaped or inverted U-shaped volume have been empirically documented, it is difficult to “rationalize” them on the basis of traders’ rebalancing and hedging needs. It is also empirically well known that changes in traders’ beliefs have a large impact on volume, as evidenced by the high correlation between volume and information flows in markets (Bessembinder et al 1996). To understand the relationship between volume dynamics and information flows and study whether or not the observed volume patterns are symptoms of excessive trading, we need an economic model that predicts (i) what trading volume should be and (ii) under what conditions volume can become excessive. There are not many such models – models of excessive trading – because it is hard enough to construct a model of trading in a dynamic environment, in which traders’ beliefs and volume are jointly determined. We aim to fill this gap by presenting a model of excessive trading based on a dynamic and local learning.

We present a dynamic equilibrium model, in which trading is locally intermediated in a large population and information aggregation occurs through this local trading process. We analyze how dispersed information is aggregated through local learning, and how this form of learning can make trading excessive. The model has a continuum of traders with the initially heterogenous asset positions subject to inventory costs. The first best allocation is achieved perfectly smoothing traders’ asset positions. The model has two frictions. First, trading is constrained to be among finite number of traders who are randomly drawn from the population distribution at each time. This precludes perfect smoothing by one round of trading, and makes many rounds of trading necessary to achieve the first best allocation. We call this local trading friction. Second, the asset position is privately known, and information
about the value of the asset is dispersed among traders. This information friction creates a speculation motive for trading. It also creates non-trivial learning dynamics, because prices in each local trading only partially (and locally) reveal private information held by the other traders. Combined together, the two frictions create an endogenous distribution of asset positions and beliefs at the aggregate level, which is subject to finite sample errors introduced at the local level. Both the initial asset position and noise in private information are assumed to be idiosyncratic, and trading averages these idiosyncratic shocks. However, the local trading friction prevents the Law of Large Numbers from eliminating the idiosyncratic shocks. This leaves locally shared information noisy, and the magnitude of the noise depends on the balance between the two trading motives: the more speculation-driven trading makes the noise small, while the more smoothing-driven trading makes it large. Through the random matching process, the small sample errors at the local level affect the joint distribution of beliefs and asset position in the entire population. This makes aggregate market conditions depend on the balance between two trading motives and how well information is shared at the local level. Conversely, the anticipated future market conditions affect the balance between two trading motives. For example, if traders anticipate that smoothing can be done more effectively at the later rounds of trading where information becomes more symmetric, traders may find it easier to speculate now without worrying about the short-run consequence for their asset positions. This can make the impact of noise in private information more important. We explicitly model this dynamic interaction between small sample errors at the local level and the evolution of aggregate market conditions, and study its implications for the trading activity and welfare.

The model shows that in the absence of the information friction the allocation monotonically approaches the first best allocation over time, and the inefficiency due to the local trading friction quickly disappears. On the other hand, with private information, richer patterns of dynamic trading can emerge due to local learning. In particular, trading can be excessive for some period in that the allocation moves away from the first best allocation.
For an asset subject to the high level of information asymmetry, both U- and inverted-U patterns of trading activity are possible, and this fluctuation in trading activity is amplified when the expected number of trading rounds increases. Thus, the model provides an explanation for the non-monotonic trading patterns based on a rational but local learning process. In the model, such trading patterns are in fact symptoms of excessive trading.

The model starts with the exogenous initial cross-sectional dispersion of the asset positions, and endogenously generates the dynamic of dispersion. Figure 1 shows how the dispersion of the asset positions changes over time for a particular parameter configuration. In the absence of the information friction, the figure would show a line monotonically approaching zero. The figure shows a case with the information friction, where there is a period in which the asset allocation becomes more dispersed.

Figure 1. The cross-sectional dispersion of the asset allocation over time.

Notes. The number of traders in each local trading is $n+1 = 4$. The number of maximum trading rounds is $T = 10$. Larger gamma means that the expected number of trading rounds is closer to $T$.

The intuition behind the pattern in Figure 1 is as follows. At the beginning of trading, the initial dispersion of asset positions is high, and prices are very noisy. Therefore, learning is slow, the impact of small sample errors on the allocation is small, and the dispersion goes down. In this initial phase ($t = 1, 2, 3$ in Figure 1), the smoothing motive is the main driver of volume dynamics. When the dispersion of asset positions goes down, prices become
a more efficient aggregator of private information. This accelerates learning, the impact of small sample errors on the allocation becomes large, and the dispersion goes up. In this phase \((t = 4, 5, 6)\), the speculation motive becomes more important overtime. Once traders accumulate sufficiently accurate information about the asset, the dispersion goes down as in the symmetric information case. This is the final phase \((t = 7, 8, 9, 10)\) where smoothing is again the dominant driver. Thus, the model predicts that trading is excessive when traders learn very quickly, but the consequence of excessive trading is undone in the subsequent trading rounds.

The model has a fixed number \(T\) of maximum trading rounds, but in each period trading ends with probability \(1 - \gamma\). Thus, \(\gamma\) captures the expected frequency of trading. The non-monotonic pattern of the dispersion shown in Figure 1 can arise even when traders are myopic in that they believe (subjectively) that each trading round is the final round (low expected frequency). This corresponds to a dotted line with \("\text{gamma} = 0"\). Figure 1 shows that as \(\gamma\) increases (higher expected frequency), the fluctuation of the dispersion is amplified. This is because the dynamic consideration can compound the feedback process of small sample errors: if traders anticipate more trading rounds in the future, they fear less about ending up with the high level of inventory, and hence trade more on private information. This affects the extent of information sharing in a local trading. The model connects many measures of trading activity (e.g. trading volume, price volatility, serial correlation) to the dispersion of the allocation. Therefore, the model identifies learning through local trading combined with the high expected frequency of trading as a source of dynamic fluctuations in trading activity.

**Related literature.**

As in many learning models, our model creates persistence in trading activity through the Bayesian updating process. Besselbinder et al (1996) document an inverted-U pattern in volume across days in a week, while Foster and Viswanathan (1993) find the U-shaped intraday pattern in volume. These patterns are difficult to generate in a learning model,
especially without a sequence of shocks that arrive over time. Bernhardt and Miao (2004) presents a model which can explain the U-shaped pattern, but they point out that it is necessary to have sequential arrival of information to generate this pattern. In our model, there is one-shot arrival of private information at the beginning, and after that all the dynamics are generated endogenously. Therefore, our model provides a potential explanation for the observed non-monotonic patterns in trading activity without relying on the sequential exogenous shocks.

This paper contributes to the literature on information aggregation with equilibrium dynamics. Duffie, Malamud and Manso (2009, 2013) study the information dynamics in a search model. Because their models are constructed to focus on information dynamics, implications for trading activity are limited. Also, in their model agents perfectly share their information when they are matched, while in our model the information asymmetry between traders remains after trading. Ostrovsky (2012) and Iyer, Johari, and Moallemi (2011) study dynamic information aggregation among finite number of traders. While they focus on public learning where trading outcome is observed by everyone, we study local learning where a trading outcome is known only to the small party directly involved in the trading. Our model builds on the idea that local interaction can introduce noise in the aggregate condition. We believe our modeling approach is useful for the analysis of decentralized markets characterized by limited public information dissemination. Amador and Weill (2012) study a learning dynamics by local interactions similar to our model, but there is no trading in their model. Our model has explicit asset trading, which allows us to study the interaction between learning dynamics, trading activity, and allocational efficiency.

The most closely related work is Golosov, Lorenzoni, and Tsyvinski (2013). They study a model of dynamic asset trading with asymmetric information, in which traders have two motives for trading as in our model. Their model has an infinite time horizon, and they focus on the long-run consequence of the one-sided learning. Our model has a finite time horizon. This allows us to focus on the short-run consequence of multi-lateral learning and
to deliver implications for dynamic patterns of trading.

This paper is organized as follows. Section 2 describes an economic environment, a trading rule, and a solution concept. We also provide two benchmark: static equilibrium and dynamic equilibrium without the information friction. Section 3 studies dynamic information aggregation with local learning. Section 4 summarizes empirical implications of the model. Section 5 concludes.

2 Model

First we describe a model environment, a trading rule, and a dynamic solution concept in the next three subsections. Second, we present two benchmarks, a static benchmark and a dynamic benchmark without the information friction in separate subsections.

2.1 Environment

The model economy has two assets and a continuum of risk neutral traders. A risky asset ("tree") has uncertain payoff. There is a convex cost of holding inventory of trees. This creates potential gains from reallocating (smoothing) the tree positions by using a non-risky asset ("money") as a means of exchange. Before a trading process starts, each trader receives the different amount of the tree and an independent noisy signal about the value of the tree. Thus, each trader has two pieces of private information: the value and position of the tree.

The trading is locally intermediated in the following sense. There is a continuum of locations. In each location, \(n+1\) traders are randomly drawn from the population distribution at each time. A finite number \(n \geq 1\) is fixed and measures the degree of the local trading friction. Note that \(n = 1\) means a bilateral trading. Each trader submits his demand for the tree, which is explicitly conditioned on the price he must pay (the amount of money exchanged for one tree). In each location, given \(n + 1\) submitted orders, a price for each trader is determined such that a market is “locally cleared” for \(n + 1\) traders. This local
trading occurs in each location. In each period, the game ends with probability \( \gamma \). When the game does not end, each trader is matched with a new \( n \) traders randomly drawn from the population. As the trading process continues, the distribution of the traders’ tree positions and their beliefs about the value of the tree in the entire economy both endogenously change. Therefore, the distribution of the type of traders endogenously changes over time. Each trader rationally forms an expectation about how the distribution evolves over time, and knows that he or she will trade with a random sample of \( n \) traders whose types are randomly drawn from the endogenously changing distribution.

There is a measure one of continuum of traders indexed by \( i \in I \). All traders have the same preference and trade a risky asset (“tree”) with the uncertain unit payoff \( v \) in exchange for a non-risky asset (“money”). The payoff \( v \) is realized at time \( t = T + 1 \) and not known to anyone until then. Traders have a common prior that

\[
v = \sqrt{\rho} v_A + \sqrt{1 - \rho} v_B
\]

and \( v_A \) and \( v_B \) are independently drawn from a normal distribution with mean zero and variance \( \tau_v^{-1} \). Each trader has two types of private information: (i) endowment of the tree \( x_{i0} \), and (ii) a private signal \( s_{i0} \) about \( v_A \). A parameter \( \rho \in [0, 1] \) controls the degree of information asymmetry without affecting the ex ante variance of \( v \) such that \( \rho = 0 \) captures a symmetric information case. The sum of endowments is the total amount of the tree in the economy. Each trader’s endowment is a realization of an independent normal random variable with mean zero and variance \( \tau_x^{-1} \). The private signal takes the form \( s_{i0} = v_A + \varepsilon_i \), where \( \varepsilon_i \) is unobserved noise in the signal, and follows a normal distribution with mean zero and variance \( \tau_\varepsilon^{-1} \). This means that \( Corr [s_{i0}, v] = \frac{\rho \tau_\varepsilon}{\sqrt{\tau_v \tau_x}} \). To summarize, random variables \( v_A, v_B, \{x_{i0}, \varepsilon_i\}_{i \in I} \) are assumed to be normally and independently distributed with zero means, and variances

\[
Var (v_A) = Var (v_B) = \tau_v^{-1}, \ Var (x_{i0}) = \tau_x^{-1}, \ Var (\varepsilon_i) = \tau_\varepsilon^{-1}.
\]
Let $b_{i0}$ be trader $i$’s initial money position. We assume that the net return on money is zero. Given an initial position $(x_{i0}, b_{i0})$ of the tree and the money, the payoff from adding $q_i$ units of the tree and $r_i$ units of the money is

$$\pi_i(q_i, r_i; x_{i0}, b_{i0}) = v(q_i + x_{i0}) + r_i + b_{i0}.$$ 

We call $(q_i, r_i)$ a trade for trader $i$. We assume that traders are risk neutral, but must incur a non-negative cost for holding a non-zero tree position:

$$C_i(q_i + x_{i0}) = \frac{\kappa}{2} (q_i + x_{i0})^2,$$

where $\kappa > 0$ measures the holding cost of the tree position. Thus, the expected utility from a trade $(q_i, r_i)$ is

$$u_i(\pi_i) = E_i[\pi_i] - \frac{\kappa}{2} (q_i + x_{i0})^2,$$

where $E_i[\cdot]$ denotes trader $i$’s conditional expectation.\(^1\)

From (1), utility is transferable via money. Hence, we focus on the efficiency of the tree allocation. The second term in (1) combined with the exogenous initial dispersion of tree positions $x_{i0}$ creates gains from trade. It also implies that the first best allocation is the perfect smoothing of tree positions. On the other hand, the dispersion in beliefs $E_i[\pi_i]$ does not directly affect social welfare.

### 2.2 Trading rule

A trading rule we study is a variation of the order-submission game studied in Kyle (1989).\(^2\)

We use a trading rule that induces a price-taking behavior to keep the dynamic analysis tractable. To do this, we need to use money allocations $\{r_i\}_{i=1}^{n+1}$. Trader $i$’s money trade is

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\(^1\)Our model environment is similar to Vives (2011) and Rostek and Weretka (2012), but it is simplified for tractability in a dynamic setup.

\(^2\)Bernhardt, Seiler and Taub (2010) and Rostek and Weretka (2011) also present a dynamic analysis of order-submission markets. However, neither studies local trading.
determined by
\[ r_i = - \left( p_i q_i + \frac{c}{2} q_i^2 \right), \]
where \( p_i \) is the unit price of the tree charged for trader \( i \) and the term \( \frac{c}{2} q_i^2 \) allows either volume tax (\( c > 0 \)) or subsidy (\( c < 0 \)). Because we do not impose any restriction on the money position, the money endowment \( \{ b_i \}_{i=1}^{n+1} \) is not important for our analysis. An individual price \( p_i \) is determined by the following pricing rule
\[ \sum_{j \neq i} q_j (p_i) = 0, \] (2)
where \( \{ q_i(\cdot) \}_{i=1}^{n+1} \) is price-contingent orders submitted by traders. Hence, trader \( i \)'s unit price \( p_i \) is determined independent of his order \( q_i(\cdot) \), while his tree allocation is determined by \( q_i(p_i) \). This makes each trader a price-taker, but internalize informational contents of prices by best-responding to every possible realization of \( p_i \). Due to the ex ante symmetry among traders, prices \( \{ p_i \}_{i=1}^{n+1} \) determined by (2) satisfy the local market clearing \( \sum_{i=1}^{n+1} q_i (p_i) = 0 \) in a symmetric equilibrium we study.

**Remark on the trading rule.**

This rule captures the idea that a market maker can offer different prices for different traders to “locally clear” a market he is making, while allowing traders to adjust the quantity traded at the offered prices. As a model of local intermediation, it seems reasonable to assume that prices can be set individually. One interpretation of the pricing rule (2) is that, the intermediary offers trader \( i \) a hypothetical price at which the other \( n \) traders would happily trade the tree among them. This price offer provides a useful information to trader \( i \), which he internalizes in his order. Because every trader does the same reasoning, there will be information aggregation in equilibrium. Therefore, this trading rule also captures the idea that intermediaries facilitate information sharing by using the orders they receive.
2.3 Dynamic equilibrium

We denote each trader’s tree trade at time $t$ by $q_{it}$ and let $r_{it} = -p_{it}q_{it}$ be the associated money trade. A trader $i$’s value of the position at the end of time $t$ is $\pi_{it} = v \left( \sum_{s=1}^{t} q_{is} + x_{i0} \right) + \sum_{s=1}^{t} r_{is} + b_{i0}$. At the end of each trading round $t \leq T - 1$, there is positive probability $1 - \gamma \in (0, 1)$ that the game ends and no more trading is allowed. If the game ends after trading at time $t$, the trader $i$’s expected utility is

$$U_{it} = E[v | F_{it}] \left( \sum_{s=1}^{t} q_{is} + x_{i0} \right) + \sum_{s=1}^{t} r_{is} + b_{i0} - \frac{\kappa}{2} \left( \sum_{s=1}^{t} q_{is} + x_{i0} \right)^2. \quad (3)$$

The information set $F_{it}$ expands over time reflecting the new information that trader $i$ learns from trading over time. The expected lifetime utility evaluated at time $t$ is defined by

$$V_{it} \equiv \frac{1 - \gamma}{1 - \gamma^{T-t+1}} E \left[ \sum_{s=t}^{T} \gamma^{s-t} U_{is} | F_{it} \right] \quad (4)$$

$$= \begin{cases} (1 - \gamma)U_{it} + \gamma E[V_{it+1} | F_{it}] & \text{for } t \leq T - 1 \\ U_{iT} & \text{for } t = T \end{cases}.$$

The parameter $\gamma$ is a probability weight put on future trading rounds, and measures how forward-looking traders are. If $\gamma = 0$, traders become myopic in that they care only period payoff (3) because they subjectively believe there is no more trading round. An alternative interpretation is that $1 - \gamma$ is a probability of aggregate event that forces traders to consume their positions.

The trading at each period is locally intermediated as described in the previous section. At each time $t = 1, \ldots, T$, given that the game has not ended, each trader is matched with a finite number $n \in \mathbb{N}$ of other traders randomly drawn from the population.

Each trader submits his order $q_{it} (\cdot ; F_{it})$ to a market maker, which is explicitly conditioned on the unit price $p_{it}$ he must pay. Given submitted $n + 1$ orders $\{q_{it} (\cdot ; F_{it})\}_{i=1}^{n+1}$, the market
maker determines \( \{p_{it}\}_{i=1}^{n+1} \) subject to the constraint

\[
\sum_{i=1}^{n+1} q_{it}(p_{it}; F_{jt}) = 0. \tag{5}
\]

Each trader \( i \) is charged \( p_{it} \) which satisfies

\[
\sum_{j \neq i} q_{jt}(p_{it}; F_{jt}) = 0. \tag{6}
\]

That is, the trader \( i \) pays the customized unit price \( p_{it} \) to the market maker. The price \( \{p_{it}\}_{i=1}^{n+1} \) is constructed as a hypothetical market-clearing price for the \( n \) orders from the other traders. The tree is allocated according to \( \{q_{it}(p_{it}; F_{it})\}_{i=1}^{n+1} \). The monetary payment is \( r_{it} = -p_{it}q_{it}(p_{it}; F_{it}) \).

In the next trading round, each trader is randomly matched with another \( n \) traders. Following the literature on dynamic matching and trading, we assume that no trader is matched with the same trader more than once regardless of the size of local trading \( n + 1 \). This captures anonymity in the large (continuum) population.

**Definition** A dynamic equilibrium is a collection \( \{q_{it}(\cdot; F_{it}), p_{it}, F_{it}\}_{i \in I; t=1\ldots T} \) which satisfy for \( t = 1, \ldots, T \),

(i) For all \( i \in I \), \( q_{it}(\cdot; F_{it}) = \arg \max_{q_{it}} V_{it} \), where \( F_{it} = F_{it-1} \cup \{p_{it}, x_{it-1}\} \)

and \( F_{i0} = \{s_{i0}, x_{i0}\} \).

(ii) For each local trading, (6) determines \( p_{it} \) for \( i = 1, \ldots, n + 1 \).

(iii) Each trader forms a Bayesian belief about the distributions of \( v \) and \( \{p_{it}, x_{it-1}\}_{i \in I; t=1\ldots T} \)

consistent with the other equilibrium variables.

In the model, there is once and for all exogenous information arrival at time 0. At time \( t \), each trader’s information set \( F_{it} \) contains the initial private information \( (s_{i0}, x_{i0}) \) and other information obtained in the past trading rounds \( s \leq t - 1 \). In the equilibrium definition
above, the information set at time $t$ also contains the price he pays at time $t$ (i.e., $p_{it} \in \mathcal{F}_{it}$) even though each trader does not know the realization of $p_{it}$ when he submits his order at time $t$. Since the order at time $t$ is allowed to be conditioned on $p_{it}$, each trader can choose his best response for each realization of $p_{it}$.

### 2.4 Static benchmark

Before studying dynamic trading, we characterize the static equilibrium allocation. First, conjecture the order of the form:

$$q_i(p, s_{i0}, x_{i0}) = \beta_s s_{i0} - \beta_x x_{i0} - \beta_p p.$$  

A key feature of this trading rule is that prices take the following form:

$$\forall i, \frac{\partial p_i}{\partial s_i} = \frac{1}{n} \sum_{j \neq i} \left( s_{j0} - \beta_x x_{j0} \right).$$  

(7)

Trader $i$ learns from equilibrium price (7), which is noisy for two reasons. First, noise in the signals $\{s_{j0}\}_{j \neq i}$ tends to offset each other, but does not disappear because $n$ is finite. Second, $\frac{1}{n} \sum_{j \neq i} x_{j0}$ is also stochastic from trader $i$’s point of view. Importantly, the impact of the second source relative to the first source is determined endogenously by trading behavior $\frac{\beta_x}{\beta_s}$. In a dynamic case, the distribution of two types of noise is also endogenous. By Bayes rule, the informational content of price (7) is

$$\left( V \left[ \frac{v}{\beta_s p_i} \right] \right)^{-1} = n\tau \left\{ 1 + \left( \frac{\beta_x}{\beta_s} \right)^2 \frac{\tau}{\tau_x} \right\}^{-1} \equiv n\tau \varphi.$$  

The variable $\varphi$ that depends on trading behavior $\frac{\beta_x}{\beta_s}$ characterizes the share of the information that trader $i$ learns from $n$ signals held by the other traders he is matched with.

Details of the characterization are gathered in the appendix. We summarize the static
results below. We use the notation $\bar{s}_{0,-i} = \frac{1}{n} \sum_{j \neq i} s_{0}$ etc to denote the average except trader $i$, and $\bar{s}_0 = \frac{1}{n+1} \sum_{i=1}^{n+1} s_{i0}$ etc for the average in each location.

**Lemma 1**

*Index equilibrium tree trade $q_i^c$ by the level of volume tax $c \in \mathbb{R}$.***

(a) For $c > -\rho$, there is an equilibrium with

$$q_i^c = \frac{\kappa}{\kappa + c} \frac{n + 1}{n} \left\{ \frac{\sqrt{\rho} (s_{i0} - \bar{s}_0)}{\kappa \left( \frac{\tau_e}{\tau_e} + 1 + n \varphi^* \right)} + \bar{x}_0 - x_{i0} \right\},$$

where $\varphi^* \in (0, 1)$ is a unique solution to

$$\frac{\varphi}{1 - \varphi} = \frac{\rho \tau_x}{\kappa^2 \tau_e} \left( \frac{\tau_u}{\tau_e} + 1 + n \varphi \right)^2.$$

(b) If $c = 0$, then the total payment by $n + 1$ traders $\sum_{i=1}^{n+1} (p_i q_i + \frac{c}{2} q_i^2)$ is negative.

(c) If $\varphi^* < \frac{n-1}{2n}$, then there exist $0 < c_b < c^m < \infty$ such that

total payment is zero at $c = c_b$ and it is uniquely maximized at $c = c^m$.

**Lemma 1** shows how finite sample bias is incorporated into the allocation. It also shows that volume tax does not affect the informational content of prices. Part (b) and (c) show that (i) without volume tax, traders as a group receive subsidy, (ii) when information aggregation is not too efficient, the budget can be balanced with volume tax.

To keep the trading dynamics as simple as possible, we set $c = 0$ and assume that the market making sector subsidizes traders for each trading round. In a dynamic case, market makers would have to charge a positive fixed fee if they were required to break even in each trading round. This does not change the trading behavior, but may affect traders’ participation decision depending on their information state.\(^3\) To balance the budget of the...
market making sector ex ante, we could assume that traders pay the fixed fee before receiving their private information. Competition among market makers will bid down the fee level so that they break even in expectation.

2.5 Dynamic benchmark without the information friction

To demonstrate that our trading rule is a reasonable one, we study a case $\rho = 0$, i.e., with symmetric information about the tree. In this case, $E[v|\mathcal{F}_it] = E[v_B|\mathcal{F}_it] = 0$ for all $t$ and $i$. The first best allocation is that everyone holds the population average tree position $E[x_{i0}] = 0$. We show that equilibrium allocation quickly converges to the first best allocation given that $n \geq 2$ and there are sufficient numbers of trading rounds.

**Lemma 2** In a dynamic equilibrium without private signals,

$$ q^*_it (p^*_it; x_{i1}) = x_{i1}; i, x_{it1}. $$

Conditional on the game not ending, the tree position at the end of time $t$ is

$$ x_{it} = \frac{1}{n^t} \sum_{j=1}^{n^t} x_{j0}, $$

and the cross-sectional distribution of the tree is $N \left( 0, n^{-t} \frac{1}{\tau_s} \right)$.

For every trading round $t$, trader $i$ is matched with $n$ new traders. In our random-matching environment with a continuum of traders, a group of traders that have traded together will never meet again. By the end of time $t$, trader $i$ has traded with $nt$ other traders, but because their sets of trading counterparties do not overlap, they indirectly exchanged their positions with $n^t$ traders. Therefore, position of the tree converges at the to pay the fixed fee. This exit behavior makes the distribution of tree positions truncated normal, and makes the analysis intractible.
rate $n^{-t}$. Compared with the efficient allocation $\pi_{t-1}^{(n+1)}$, one round of trading achieves less smoothing, since $\pi_{t-1,-i}$ is the average of $n$ traders’ positions, not $n+1$. However, as long as $n \geq 2$, dynamic trading can achieve the efficient allocation after the sufficient number of trading rounds.\footnote{When each local trading is bilateral ($n = 1$), this convergence does not occur, because two traders switch their tree positions: $\pi_{t-1,-i} = x_{jt-1}$.}

In the next section, we investigate how information aggregation and trading interact in a dynamic model.

\section{Dynamic Information Aggregation with Local Learning}

To analyze the interaction of dynamic information aggregation and allocation, we construct a dynamic equilibrium with the following three properties:

\textbf{Property I}: At the beginning of period $t$, trader $i$ has the tree position $x_{it-1}$, and the cross-sectional distribution of $x_{it-1}$ is $N \left( 0, \tau_{xt-1}^{-1} \right)$.

\textbf{Property II}: At the beginning of period $t$, trader $i$ has $t$ signals $\{s_{ik}\}_{k=0}^{t-1}$, where $s_{ik} = v_A + \varepsilon_{ik}$, $k = 0, \ldots, t-1$, and the distribution of the noise $\varepsilon_{ik}$ is $N \left( 0, \frac{1}{\tau_{\varepsilon_{n^k}} \varphi_k} \right)$ independent across traders.

\textbf{Property III}: For time $t$ trading, trader $i$ submits an order

$$q_{it}(p_{it}; \mathcal{F}_{it}) = \sum_{k=0}^{t-1} \beta_{stk} s_{ik} - \beta_{xt} x_{it-1} - \beta_{pt} p_{it}. \quad (8)$$

We proceed in two steps. First, we establish Property I and II taking Property III as given. Second, we characterize $\{ \{\beta_{stk}\}_{k=0}^{t-1}, \beta_{xt}, \beta_{pt} \}_{t=1}^{T}$ by the guess-and-verify method combined with a backward induction.
3.1 Property I and II

Property I and II are satisfied at \( t = 1 \) with \( \varphi_0 = 1 \), given our assumptions on \( \{s_{i0}, x_{i0}\}_{i \in I} \). By the end of time \( t \) trading, each trader has directly interacted with \( nt \) other traders. However, because no trader meets twice, each trader has \textit{indirectly} interacted with \( n^t \) other traders. Therefore, the most trader \( i \) can potentially learn from time \( t \) trading is the information that could be obtained from \( n^t \) independent signals \( \{s_{j0}\} \). Thus, \( \varphi_k \) in Property II measures a fraction of information each trader learns at time \( k \) relative to the potentially available information \( \tau_e \). For each \( t = 1, \ldots, T \), conjectured \( t + 2 \) coefficients \( \{ (\beta_{stk})_{k=0}^{t-1}, \beta_{xt}, \beta_{pt} \} \) must be verified and characterized. Also, signals \( \{s_{ik}\}_{k=1}^{t-1} \) must be constructed from the equilibrium prices to satisfy Property II.

Given the pricing rule (6) and the conjecture (8), information learned from \( p_{it} \) is

\[
s_{it} \equiv \frac{\beta_{pt}}{\sum_{k=0}^{t-1} \beta_{stk}} p_{it} = \frac{\frac{1}{n} \sum_{j \neq k=0}^{t-1} \beta_{stk}s_{jk}}{\sum_{k=0}^{t-1} \beta_{stk}} - \frac{\beta_{xt}}{\sum_{k=0}^{t-1} \beta_{stk}} \bar{x}_{t-1, -i}.
\]

Using \( \tilde{\beta}_{stk} = \frac{\beta_{stk}}{\sum_{k=0}^{t-1} \beta_{stk}} \), \( \tilde{\beta}_{xt} = \frac{\beta_{xt}}{\sum_{k=0}^{t-1} \beta_{stk}} \) and \( \tilde{\beta}_{pt} = \frac{\beta_{pt}}{\sum_{k=0}^{t-1} \beta_{stk}} \) this can be written as

\[
s_{it} \equiv \tilde{\beta}_{pt} p_{it} \tag{9}
\]

\[
= \sum_{k=0}^{t-1} \tilde{\beta}_{stk} s_{k, -i} - \tilde{\beta}_{xt} \bar{x}_{t-1, -i}
\]

\[
= \nu_A + \sum_{k=0}^{t-1} \tilde{\beta}_{stk} x_{k, -i} - \tilde{\beta}_{xt} \bar{x}_{t-1, -i}.
\]

Suppose Property II holds at period \( t \). Then the signal \( s_{it} \) is independent across \( i \) and also independent from \( \{s_{ik}\}_{k=0}^{t-1} \) conditional on \( \nu_A \), because no pair of traders meets twice and
does not share the history before matching. Therefore,

\[
(\text{Var}[s_{it}|v_t])^{-1} = \left( \sum_{k=0}^{t-1} \beta_{sk}^2 \frac{1}{n \tau \eta k} + \beta_{xt}^2 \frac{1}{n \tau x t-1} \right)^{-1}.
\]

Then we can define \( \varphi_t \) by

\[
\frac{1}{n^{t-1} \varphi_t} = \frac{1}{n} \sum_{k=0}^{t-1} \beta_{sk}^2 + \frac{1}{n} \beta_{xt}^2 \tau_x \tau_{xt-1}.
\]

(10)

Thus, Property II holds at period \( t + 1 \) with \( s_{it} \) defined by (9) and \( \varphi_t \) defined by (10).

Note that trader \( i \)'s accumulated signals \( \{s_{ik}\}_{k=1}^{t-1} \) are informationally equivalent to his price history \( \{p_{ik}\}_{k=1}^{t-1} \).

Given the conjecture (8), prices and quantity traded in equilibrium at time \( t \) must satisfy

\[
\beta_{pt} p_{it} = \sum_{k=0}^{t-1} \beta_{st} (s_{ik} - \bar{s}_{k,-i}) - \beta_{xt} (x_{it-1} - \bar{x}_{t-1,-i}).
\]

Hence, \( x_{it-1} \) and \( x_{it} \) are related in equilibrium by the following condition:

\[
x_{it} = \sum_{k=0}^{t-1} \beta_{st} (s_{ik} - \bar{s}_{k,-i}) + \beta_{xt} \bar{x}_{t-1,-i} + (1 - \beta_{xt}) x_{it-1}.
\]

(11)

Suppose Property I and II hold at time \( t \). Given that \( x_{it-1} \) has a distribution \( N \left( 0, \tau_{xt-1}^{-1} \right) \) and \( \varepsilon_{ik} \) has \( N \left( 0, \frac{1}{\tau \eta k} \right) \) independent across \( i \) up to \( k = t-1 \), (11) implies that \( E[x_{it}] = 0 \) and

\[
V[x_{it}] = \frac{1}{n} \left( \sum_{k=0}^{t-1} \beta_{sk}^2 \tau \eta k \varphi_k \right)^2 + \sum_{k=0}^{t-1} \beta_{st}^2 \tau \eta k \varphi_k + (1 - \beta_{xt})^2 \frac{1}{\tau x t-1}.
\]

\[
= \frac{1}{\tau \eta n \varphi_t} \left( \sum_{k=0}^{t-1} \beta_{st}^2 \tau \eta k \varphi_k \right)^2 + \sum_{k=0}^{t-1} \beta_{st}^2 \tau \eta k \varphi_k + (1 - \beta_{xt})^2 \frac{1}{\tau x t-1}.
\]
Therefore,
\[
\frac{\tau_{xt}}{\tau_e} = \left( \sum_{k=0}^{t-1} \beta_{stk} \right)^2 \left( \frac{1}{n^t \phi_t} + \sum_{k=0}^{t-1} \frac{\gamma_{stk}^2}{n^k \phi_k} \right) + (1 - \beta_{xt})^2 \frac{\tau_{xt}}{\tau_{xt-1}}.
\] (12)

Thus, Property I holds at time \( t + 1 \) with \( \tau_{xt} \) determined by (12).

Two dynamic equations (10) and (12) jointly describe learning and allocation dynamics given equilibrium trading behavior characterized by \( \{ \beta_{stk} \}_{k=0}^{t-1}, \beta_{xt}, \beta_{pt} \}_{t=1}^T \). It should be clear from the derivation of these two equations that Properties I and II hold for time \( t \) if they hold for time up to \( t - 1 \). Hence, Properties I and II were verified by induction given Property III.

3.2 Property III

We verify Property III by showing that trader \( i \)'s optimal order takes the conjectured form (8) given that all the others use the same form. Each trader \( i \)'s belief about \( v_A \) at time \( t \) is summarized by its conditional mean and variance, and characterized by Bayes rule. To suppress expressions, define
\[
\chi_t \equiv \sum_{k=0}^{t} n^k \phi_k.
\]

We call \( \chi_t \) a stock information at the end of period \( t \), because it measures the cumulative amount of information each trader learned by the end of period \( t \). Note that \( \chi_t \geq 1 \) because \( \phi_0 \equiv 1 \).

\[
E_{it}[v_A] = \frac{\sum_{k=0}^{t} n^k \phi_k s_{ik}}{\frac{\tau_e}{\tau_x} + \chi_t} = \frac{1}{\frac{\tau_e}{\tau_x} + \chi_t} \left( \sum_{k=0}^{t-1} n^k \phi_k s_{ik} + n^t \phi_t \tilde{\beta}_p p_{it} \right).
\] (13)

Conditional expectation (13) is linear in \( \{ s_{ik} \}_{k=0}^{t-1} \) and \( p_{it} \). Given this belief, we derive the optimal order.

First, consider the final period \( t = T \). Because there is no more trading after period \( T \),
the optimal order takes the same form as in the static case:

\[
q_{iT}(p_{iT}; F_{iT}) = \sqrt{\rho} E_{iT}[v_A] - p_{iT} - x_{iT-1} = \frac{\sqrt{\rho}}{\kappa} \left( \frac{T^{-1}}{\kappa} \sum_{k=0}^{T-1} n^k \varphi_{k sT} \right) - x_{iT-1} - \frac{1}{\kappa} \left( 1 - \frac{\sqrt{\rho} T^{T} \varphi_T \tilde{\beta}_{pT}}{\frac{\tau_{\epsilon}}{\tau_{\epsilon}} + \chi_T} \right) p_{iT}.
\]

By equating coefficients with those in (8) for \( t = T \),

\[
\beta_{sT} = \frac{\sqrt{\rho}}{\kappa} \frac{n^k \varphi_k}{\frac{\tau_{\epsilon}}{\tau_{\epsilon}} + \chi_T}, \quad k = 0, \ldots, T - 1,
\]

\[
\beta_{xT} = 1 \quad \text{and} \quad \beta_{pT} = \frac{1}{\kappa} \left( 1 - \frac{\sqrt{\rho} T^{T} \varphi_T \tilde{\beta}_{pT}}{\frac{\tau_{\epsilon}}{\tau_{\epsilon}} + \chi_T} \right).
\]

From the expression of \( \beta_{sT} \),

\[
\sum_{k=0}^{T-1} \beta_{sT} = \frac{\sqrt{\rho}}{\kappa} \frac{\chi_{T-1}}{\frac{\tau_{\epsilon}}{\tau_{\epsilon}} + \chi_T} \quad \text{and} \quad \tilde{\beta}_{sT} = \frac{n^k \varphi_k}{\chi_{T-1}}, \quad k = 0, \ldots, T - 1,
\]

\[
\tilde{\beta}_{xT} = \frac{\kappa}{\sqrt{\rho}} \frac{\frac{\tau_{\epsilon}}{\tau_{\epsilon}} + \chi_T}{\chi_{T-1}},
\]

\[
\tilde{\beta}_{pT} = \frac{1}{\sqrt{\rho} \chi_{T-1}} \left( \frac{\frac{\tau_{\epsilon}}{\tau_{\epsilon}} + \chi_T}{\chi_{T-1}} \left( 1 - \frac{\sqrt{\rho} T^{T} \varphi_T \tilde{\beta}_{pT}}{\frac{\tau_{\epsilon}}{\tau_{\epsilon}} + \chi_T} \right) \right)
\]

\[
= \frac{1}{\sqrt{\rho} \chi_{T-1}} \left( \frac{\frac{\tau_{\epsilon}}{\tau_{\epsilon}} + \chi_T}{\chi_{T-1}} \chi_T - \sqrt{\rho} T^{T} \varphi_T \tilde{\beta}_{pT} \right).
\]

The last condition can be solved for \( \tilde{\beta}_{pT} \):

\[
\tilde{\beta}_{pT} = \frac{\frac{\tau_{\epsilon}}{\tau_{\epsilon}} + \chi_T}{\sqrt{\rho} \left( \chi_{T-1} + n^T \varphi_T \right)} = \frac{\frac{\tau_{\epsilon}}{\tau_{\epsilon}} + \chi_T}{\sqrt{\rho} \chi_T}.
\]

Hence,

\[
\beta_{pT} = \frac{1}{\kappa} \left( 1 - \sqrt{\rho} T^{T} \varphi_T \frac{\tau_{\epsilon}}{\tau_{\epsilon}} + \chi_T \right) = \frac{1}{\rho / \chi_T} \chi_{T-1}.
\]
For $t = T$, this verifies Property III and

$$q_{iT}(p_{iT}; F_{iT}) = \frac{\sqrt{\rho}}{\kappa (\frac{\tau_u}{\tau_e} + \chi_T)} \sum_{k=0}^{T-1} n^k \varphi_k s_{ik} - x_{iT-1} - \frac{1}{\kappa} \frac{\chi_{T-1}}{\chi_T} p_{iT},$$

$$p^*_{iT} = \sum_{k=0}^{T-1} \frac{\beta_s T_k}{\beta_{pT}} s_{k,-i} - \frac{\beta_s T_i}{\beta_{pT}} x_{iT-1,-i}$$

$$= \frac{\sqrt{\rho} \chi_T}{(\frac{\tau_u}{\tau_e} + \chi_T)} \sum_{k=0}^{T-1} n^k \varphi_k s_{k,-i} - \frac{\kappa \chi_T}{\chi_T} x_{iT-1,-i},$$

$$q_{iT}(p^*_{iT}; F_{iT}) = \frac{\sqrt{\rho}}{\kappa (\frac{\tau_u}{\tau_e} + \chi_T)} \sum_{k=0}^{T-1} n^k \varphi_k (s_{ik} - s_{k,-i}) - (x_{iT-1} - x_{iT-1,-i}),$$

$$x_{iT} = \frac{\sqrt{\rho}}{\kappa (\frac{\tau_u}{\tau_e} + \chi_T)} \sum_{k=0}^{T-1} n^k \varphi_k (s_{ik} - s_{k,-i}) + x_{iT-1,-i}.$$  

Substituting derived coefficients into (10) gives

$$\frac{1}{n^{T-1} \varphi_T} = \sum_{k=0}^{T-1} \left( \frac{n^k \varphi_k}{n^{k_{T-1}}} \right)^2 + \left( \frac{\kappa \frac{\tau_u}{\tau_e} + \chi_T}{\sqrt{\rho} \chi_T} \right)^2 \frac{\tau_e}{\tau_{xT-1}}$$

$$= \frac{1}{\chi_{T-1}} + \frac{\kappa^2}{\rho} \frac{\tau_e}{\tau_{xT-1}} \left( \frac{\tau_u}{\tau_e} + \chi_T \right)^2 \chi_{T-1}).$$

This can be seen as an equation in $\varphi_T$ given $(\varphi_1, \ldots, \varphi_{T-1})$ and $\tau_{xT-1}$:

$$\frac{n^{T-1} \varphi_T}{\chi_{T-1}} - 1 + \frac{\kappa^2}{\rho} \frac{\tau_e}{\tau_{xT-1}} \left( \frac{\tau_u}{\tau_e} + \chi_T \right)^2 \chi_{T-1} = 0.$$

$$\Leftrightarrow \frac{\kappa^2}{\rho} \frac{\tau_e}{\tau_{xT-1}} \left( \frac{\tau_u}{\tau_e} + \chi_T \right)^2 \chi_{T-1} = 1 - \frac{n^{T-1} \varphi_T}{\chi_{T-1}}.$$
Because the right hand side must be positive for the solution to exist,
\[
\frac{n^{T-1} \varphi_T}{1 - \frac{n^{T-1}}{\chi_T} \varphi_T} = \frac{\rho}{\kappa^2} \frac{\tau_{xT-1}}{\tau_x} \left( \frac{\chi_{T-1}}{\tau_x + \chi_T} \right)^2.
\] (17)

For a fixed \((\varphi_1, \ldots, \varphi_{T-1})\) and \(\tau_{xT-1}\), the left hand side of (17) is increasing in \(\varphi_T\) and continuously change from zero to positive infinity for \(\varphi_T \in \left[0, \frac{x_T}{nT} \right]\) and the right hand side is decreasing in \(\varphi_T\). Hence, there is a unique \(\varphi_T^*\) that solves (17) for any given \((\varphi_1, \ldots, \varphi_{T-1})\) and \(\tau_{xT-1}\).

Also, substituting derived coefficients into (12) gives
\[
\frac{\tau_x}{\tau_{xT}} = \left( \frac{\sqrt{\rho}}{\kappa} \frac{\chi_{T-1}}{\tau_x + \chi_T} \right)^2 \left( \frac{1}{nT \varphi_T} + \frac{1}{\chi_{T-1}} \right)
\[
= \frac{\rho}{\kappa^2} \left( \frac{\chi_{T-1}}{\tau_x + \chi_T} \right)^2 \left( \frac{1}{nT \varphi_T} + 1 \right)
\[
= \frac{\rho}{\kappa^2} \left( \frac{\chi_{T-1}}{\tau_x + \chi_T} \right)^2 \frac{\chi_T}{nT \varphi_T \chi_{T-1}}
\[
= \frac{\rho}{\kappa^2 nT \varphi_T} \frac{\chi_T}{\tau_x + \chi_T}.
\]

Because \(\beta_{xT} = 1\) under the current trading rule, whatever position traders have at the beginning of period \(T\) is traded away and averaged out and \(x_{iT-1}\) does not directly affect \(x_{iT}\). Hence \(\frac{\tau_x}{\tau_{xT-1}}\) does not explicitly show up in the expression of \(\frac{\tau_x}{\tau_{xT}}\) above. However, (17) shows that \(\varphi_T\) is decreasing in \(\frac{\tau_x}{\tau_{xT-1}}\). Hence, the distribution of trees in the previous period affects the distribution of the tree in the following period through the learning channel.

Next, we characterize an optimal order in period \(T - 1\). Trader \(i\) solves
\[
\max_{q_{iT-1}} (1 - \gamma) \left\{ E_{iT-1} [v] (q_{iT-1} + x_{iT-2}) - \frac{\rho}{2} (q_{iT-1} + x_{iT-2})^2 - p_{iT-1} q_{iT-1} \right\}
\[
+ \gamma E_{iT} [v] x_{iT} - \frac{\rho}{2} x_{iT}^2 - (p_{iT-1} q_{iT-1} + p_{iT} q_{iT} (p_{iT}^*)) \right\},
\]
where $p_{iT}^*, q_{iT} (p_{iT}^*)$, $x_{iT}$ are given by (14), (15), (16). The first line in the expression above is the expected utility for the case the game ends after trading in period $T - 1$, while the second line corresponds to the case where there is another trading round. First, money payment $p_{iT-1}q_{iT-1}$ is sunk and hence shows up in both lines. Second, given the optimal order in period $T$, neither $p_{iT}^*$ nor $x_{iT}$ depends on $q_{iT-1}$, while $q_{iT} (p_{iT}^*)$ does depend on $q_{iT-1}$ through the term $-x_{iT-1} = - (q_{iT-1} + x_{iT-2})$. Thus, dropping irrelevant terms, the objective can be written as

$$\max_{q_{iT-1}} (1 - \gamma) \left\{ E_{iT-1} [v] q_{iT-1} - \frac{\rho}{2} (q_{iT-1} + x_{iT-2})^2 \right\} - p_{iT-1}q_{iT-1} + \gamma E_{iT-1} [p_{iT}^*] q_{iT-1}.$$  

The last term shows up because the additional tree today would save the purchase tomorrow if and only if the game continues. Hence, the optimal order at $T - 1$ is

$$q_{iT-1} (p_{iT-1}; F_{iT-1}) = \frac{(1 - \gamma) E_{iT-1} [v] + \gamma E_{iT-1} [p_{iT}^*] - p_{iT-1}}{\kappa (1 - \gamma)} - x_{iT-2}.$$  

From (14),

$$E_{iT-1} [p_{iT}^*] = \sum_{k=0}^{T-1} \beta_s E_{iT-1} [s_{k-1-i}] = \frac{\sqrt{\rho X_T}}{\tau_e + \chi T} E_{iT-1} [v_A].$$  

Hence,

$$q_{iT-1} (p_{iT-1}; F_{iT-1}) = \frac{1}{\kappa (1 - \gamma)} \left\{ \left( 1 - \gamma + \frac{\chi T}{\tau_e + X_T} \right) \sqrt{\rho E_{iT-1} [v_A]} \right\} - x_{iT-2} = \frac{1}{\kappa} \left\{ \left( 1 - \gamma + \frac{\chi T}{\tau_e (1 - \gamma)} \right) \sqrt{\rho E_{iT-1} [v_A]} \right\} - x_{iT-2}.$$  

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Finally, recall from (13) that

$$E_{iT-1}[v_A] = \frac{1}{\tau_w/\tau_e + \chi_{T-1}} \left( \sum_{k=0}^{T-2} n_k \varphi_{k s k} + n_{T-1}^{T-1} \varphi_{T-1} \beta_{iT-1} \right).$$

Therefore,

$$q_{iT-1}(p_{iT-1}; F_{iT-1}) = \left( 1 + \frac{\gamma}{1 - \gamma} \frac{\chi_T}{\tau_w/\tau_e + \chi_T} \right) \frac{\sqrt{\rho}}{1 - \gamma} \frac{n_k \varphi_k}{\tau_w/\tau_e + \chi_{T-1}} - x_{iT-2}$$

$$- \frac{1}{\kappa} \left\{ \frac{1}{1 - \gamma} - \left( 1 + \frac{\gamma}{1 - \gamma} \frac{\chi_T}{\tau_w/\tau_e + \chi_T} \right) \frac{\sqrt{\rho} n_{T-1}^{T-1} \varphi_{T-1} \beta_{iT-1}}{\tau_w/\tau_e + \chi_{T-1}} \right\} p_{iT-1}.$$

By equating coefficients with those in (8) for $t = T - 1$,

$$\beta_{sT-1k} = \left( 1 + \frac{\gamma}{1 - \gamma} \frac{\chi_T}{\tau_w/\tau_e + \chi_T} \right) \frac{\sqrt{\rho}}{1 - \gamma} \frac{n_k \varphi_k}{\tau_w/\tau_e + \chi_{T-1}}, \quad k = 0, \ldots, T - 2,$$

$$\beta_{sT-1} = 1,$$

$$\beta_{pT-1} = \frac{1}{\kappa} \left\{ \frac{1}{1 - \gamma} - \left( 1 + \frac{\gamma}{1 - \gamma} \frac{\chi_T}{\tau_w/\tau_e + \chi_T} \right) \frac{\sqrt{\rho} n_{T-1}^{T-1} \varphi_{T-1} \beta_{pT-1}}{\tau_w/\tau_e + \chi_{T-1}} \right\}.$$

From the expression of $\beta_{sT-1k},$

$$\sum_{k=0}^{T-2} \beta_{sT-1k} = \left( 1 + \frac{\gamma}{1 - \gamma} \frac{\chi_T}{\tau_w/\tau_e + \chi_T} \right) \frac{\sqrt{\rho}}{1 - \gamma} \frac{\chi_{T-2}}{\tau_w/\tau_e + \chi_{T-1}},$$

$$\tilde{\beta}_{sT-1k} = \frac{n_k \varphi_k}{\chi_{T-2}}, \quad k = 0, \ldots, T - 2,$$

$$\tilde{\beta}_{sT-1} = \left( 1 + \frac{\gamma}{1 - \gamma} \frac{\chi_T}{\tau_w/\tau_e + \chi_T} \right)^{-1} \frac{\kappa}{\sqrt{\rho}} \frac{\chi_T}{\tau_w/\tau_e + \chi_{T-1}},$$

$$\beta_{pT-1} = \frac{1}{\kappa} \left\{ \frac{1}{1 - \gamma} - \sqrt{\rho} n_{T-1}^{T-1} \varphi_{T-1} \beta_{iT-1} \right\}^{-1} \frac{\sqrt{\rho}}{\kappa} \frac{\chi_{T-2}}{\tau_w/\tau_e + \chi_{T-1}} \beta_{pT-1},$$

$$= \frac{1}{\kappa} \left\{ \frac{1}{1 - \gamma} - \kappa \frac{n_{T-1}^{T-1}}{\chi_{T-2}} \beta_{pT-1} \right\}.$$
The last condition can be solved for $\beta_{\rho T-1}$:

$$\beta_{\rho T-1} = \frac{1}{(1-\gamma) \kappa} \left( 1 + \frac{n^{T-1} \varphi_{T-1}}{\chi_{T-2}} \right)^{-1}$$

$$= \frac{1}{(1-\gamma) \kappa} \chi_{T-2}.$$  

We define $\Gamma_1 \equiv 1 + \frac{\gamma}{1-\gamma} \frac{\chi_T}{\chi_T}$ to write

$$q_{iT-1}(p_{iT-1}; F_{iT-1}) = \Gamma_1 \sqrt{\rho} \sum_{k=0}^{T-2} n^k \varphi_{sik}$$

$$= \frac{1}{\rho \tau_x + \chi_{T-1}} - x_{iT-2} - \frac{1}{(1-\gamma) \rho \chi_{T-1}} p_{iT-1}.$$ 

Note that $\Gamma_1 \in \left(1, \frac{1}{1-\gamma}\right)$.

Proceeding similarly as before,

$$\frac{1}{n^{T-2} \varphi_{T-1}} = \frac{1}{\chi_{T-2}} + \rho \tau_x + \chi_{T-1}$$

$$= \frac{1}{\chi_{T-2}} + \frac{\kappa^2}{\rho \tau_x + \chi_{T-1}} \left( \frac{1}{\Gamma_1} \frac{\tau_x + \chi_{T-1}}{\chi_{T-2}} \right)^2.$$ 

This can be seen as an equation in $\varphi_{T-1}$ given $(\varphi_1, \ldots, \varphi_{T-2}, \varphi_T)$ and $\tau_{xT-2}$:

$$\frac{n^{T-2} \varphi_{T-1}}{\chi_{T-2}} - 1 + \frac{\kappa^2}{\rho \tau_x + \chi_{T-1}} \left( \frac{1}{\Gamma_1} \frac{\tau_x + \chi_{T-1}}{\chi_{T-2}} \right)^2 n^{T-2} \varphi_{T-1} = 0.$$ 

$$\Leftrightarrow \frac{\kappa^2}{\rho \tau_x + \chi_{T-1}} \left( \frac{1}{\Gamma_1} \frac{\tau_x + \chi_{T-1}}{\chi_{T-2}} \right)^2 n^{T-2} \varphi_{T-1} = 1 - n^{T-2} \varphi_{T-1}.$$ 

Because the right hand side must be positive for the solution to exist,

$$\frac{n^{T-2} \varphi_{T-1}}{1 - n^{T-2} \varphi_{T-1}} = \frac{\rho \tau_{xT-2}}{\kappa^2 \tau_x} \left( \Gamma_1 \frac{\chi_{T-2}}{\tau_x + \chi_{T-1}} \right)^2.$$  

(18)

Given $(\varphi_1, \ldots, \varphi_{T-2})$, the left hand side of (18) is increasing in $\varphi_{T-1}$ and continuously change
from zero to positive infinity for $\varphi_{T-1} \in \left[0, \frac{\chi_{T-2}}{\tau_{T-1}} \right]$. On the other hand, the right hand side is decreasing in $\varphi_{T-1}$ unless $\gamma$ is too large.\textsuperscript{5} We focus on the case where there is a unique $\varphi_{T-1}^*$ that solves (18) for any given $(\varphi_1, \ldots, \varphi_{T-2}, \varphi_T)$ and $\tau_{xT-2}$.

Also, substituting derived coefficients into (12) gives

$$\frac{\tau_{xT-1}}{\tau_{T-1}} = \left( \Gamma_1 \frac{\sqrt{\rho}}{\kappa} \frac{\tau_{xT-2}}{\chi_{T-1}} \right)^2 \left( \frac{1}{n^{T-1} \varphi_{T-1}} + \sum_{k=0}^{T-2} \left( \frac{n^k \varphi_k}{\chi_{T-2}} \right)^2 \right)$$

$$= \frac{\rho}{\kappa^2} \left( \Gamma_1 \frac{\tau_{xT-2}}{\chi_{T-1}} \right)^2 \left( \frac{1}{n^{T-1} \varphi_{T-1}} + \frac{1}{\chi_{T-2}} \right)$$

$$= \frac{\rho}{\kappa^2} \left( \Gamma_1 \frac{\tau_{xT-2}}{\chi_{T-1}} \right)^2 \frac{\chi_{T-1}}{n^{T-1} \varphi_{T-1} \chi_{T-2}}$$

$$= \frac{\rho \chi_{T-2} \chi_{T-1}}{\kappa^2 n^{T-1} \varphi_{T-1}} \left( \frac{\Gamma_1}{\chi_{T-1}} \right)^2.$$ 

This last expression can be substituted into (17) to obtain

$$\frac{n^{T-1} \varphi_T}{1 - \frac{n^{T-1}}{\chi_{T-1}} \varphi_T} = \left( \frac{\tau_{xT-2}}{\chi_{T-1}} + \frac{\chi_{T-1}}{\Gamma_1} \right)^2 \left( \frac{\chi_{T-1}}{\chi_{T-2} \chi_{T-1}} \right)^2.$$

$\Leftrightarrow \varphi_T = \frac{1}{\chi_{T-2}} \left( \chi_{T-1} - n^{T-1} \varphi_T \right) \left( \frac{1}{\Gamma_1} \frac{\tau_{xT-2}}{\tau_{T-1}} + \frac{\chi_{T-1}}{\Gamma_1} \frac{\chi_{T-1}}{\chi_{T-2}} \right)^2 \varphi_{T-1}$. 

$\Leftrightarrow \varphi_T = \left( 1 + \frac{n^{T-1}}{\chi_{T-2}} \left( \varphi_{T-1} - \varphi_T \right) \right) \left( \frac{1}{\Gamma_1} \frac{\tau_{xT-2}}{\tau_{T-1}} + \frac{\chi_{T-1}}{\Gamma_1} \frac{\chi_{T-1}}{\chi_{T-2}} \right)^2 \varphi_{T-1}$. 

Because $\frac{1}{\Gamma_1} \frac{\tau_{xT-2}}{\tau_{T-1}} + \frac{\chi_{T-1}}{\chi_{T-2}} < 1$, this shows that $\varphi_T < \varphi_{T-1}$ in equilibrium.

The rest of the characterization of the dynamic equilibrium uses an induction argument. We define a sequence $\{\Gamma_{T-t}\}_{t=1}^T = \{\Gamma_{T-1}, \Gamma_{T-2}, \ldots, \Gamma_1, \Gamma_0\}$ as follows. First, $\Gamma_0 \equiv 1$. For $\textsuperscript{5}$\(\Gamma_1\) as a function of $\varphi_1, \ldots, \varphi_T$ is increasing in each argument if $\gamma > 0$, but $\frac{\tau_{xT-1}}{\tau_{T-1}} \gamma = \frac{\chi_{T-1}}{\chi_{T-2}} \left( \frac{\tau_{xT}}{\chi_{T}} \right) \left( \frac{\tau_{xT}}{\chi_{T}} \right)$ is decreasing in $\varphi_{T-1}$ if $\gamma \leq \frac{1}{2}$.
\[ t = 1, \ldots, T - 1, \]
\[ \Gamma_{T-t} \equiv 1 + \frac{\gamma}{1 - \gamma 1 \{ t = T - 1 \}} \frac{\chi_{t+1}}{\tau_{e} + \chi_{t+1}} \Gamma_{T-(t+1)}. \]  
\[ (19) \]

**Proposition** For \( t = 1, \ldots, T \),

(a) \( \beta_{stk} = \frac{\sqrt{p}}{\kappa} \frac{\Gamma_{T-t}}{\tau_{e} + \chi_{t}} n^{k} \varphi_{k} \) for \( k = 1, \ldots, t - 1 \), \( \beta_{xt} = 1 \), and \( \beta_{pt} = \frac{1}{\kappa (1 - \gamma 1 \{ t < T \})} \frac{\chi_{t-1}}{\chi_{t}}. \)

(b) For \( t = 1 \), \( \varphi_{1} \) solves

\[ \frac{\varphi_{1}}{1 - \varphi_{1}} = \frac{\rho \tau_{x}}{\kappa^{2} \tau_{e}} \left( \frac{\Gamma_{T-1}}{\Gamma_{T-(t-1)}} \frac{\tau_{e}}{\tau_{e} + \chi_{t}} \right)^{2}. \]  
\[ (20) \]

For \( t = 2, \ldots, T \), \( \varphi_{t} \) solves

\[ \varphi_{t} = \left( 1 + \frac{n^{t-1}}{\chi_{t-2}} (\varphi_{t-1} - \varphi_{t}) \right) \left( \frac{\Gamma_{T-t}}{\Gamma_{T-(t-1)}} \frac{\tau_{e}}{\tau_{e} + \chi_{t}} \right)^{2} \varphi_{t-1}. \]  
\[ (21) \]

(c) The distribution of the tree at the end of time \( t \) is \( N(0, \tau_{xt}^{-1}) \), where

\[ \frac{1}{\tau_{xt}} = \frac{1}{n - \frac{n^{t}}{\chi_{t-1}} \varphi_{t} \tau_{xt-1}}. \]

(d) \( \frac{\text{Var}[x_{t}]}{\text{Var}[x_{t-1}]} \leq 1 \iff n^{t} \varphi_{t} \in \left( 0, \frac{n-1}{2} \chi_{t-1} \right] \]

**Proposition (a)** shows that larger \( \gamma \) raises \( \Gamma_{T-t} \) and makes trading more speculation-driven. Note that \( \{ \Gamma_{T-t} \}_{t=1}^{T-1} \) may depend on \( \{ \varphi_{t} \}_{t=1}^{T} \), but when they do they are continuous in \( \{ \varphi_{t} \}_{t=1}^{T} \) and bounded in \( \left( 1, \frac{1}{1 - \gamma} \right) \). Therefore, as (b) shows, (20) and (21) jointly define a continuous mapping from \( \mathbb{R}^{T} \) into itself, whose fixed point characterizes the equilibrium value of \( \{ \varphi_{t} \}_{t=1}^{T} \). For any \( T \), a fixed point exists since \( \varphi_{t} \in [0, \overline{\varphi}] \) is bounded.\(^6\) However, it must be solved numerically.

**Proposition (c)** shows that the change in the dispersion of the allocation \( \frac{1 + n^{t}}{n \chi_{t-1}} \varphi_{t} \) is bounded below by \( \frac{1}{n} \) and decreasing in \( \varphi_{t} \). As information aggregation slows down (\( \varphi_{t} \to 0 \)),

\(^6\)The upper bound \( \overline{\varphi} \) is provided by the following recursive relationship: \( \overline{\varphi}_{t+1} = \frac{1}{n} \sum_{k=0}^{t} n^{k} \overline{\varphi}_{k} \) with \( \overline{\varphi}_{1} \equiv 1 \).
the variance ratio becomes smaller and approaches the benchmark case with no information aggregation $\frac{1}{n}$. The lower bound $\frac{1}{n}$ is attained by the case with symmetric information or $\varphi_t \to 0$. **Proposition (d)** shows that the dispersion can increase, and it happens if and only if the amount of information aggregation in that period $n^t \varphi_t$ exceeds the bar set by the information learned so far, $\frac{n-1}{2} \chi_{t-1}$.

Next, define $K_t \equiv \frac{\Gamma_{T-t}}{\Gamma_{T-(t-1)}}, \frac{\tau_v + \chi_{t-1}}{\tau_v + \chi_t}$.

**Lemma 3 (learning speed $\frac{\varphi_t}{\varphi_{t-1}}$)**

(a) If $\gamma \tau_v = 0$, then $\varphi_1 = \varphi^* > \varphi_2 > .. > \varphi_T$.

(b) If $\gamma \tau_v > 0$, then $\frac{\varphi_T}{\varphi_{T-1}} < 1$ and for each $t = 2, .., T - 1$, only one of the following three cases is possible:

(i) $1 < \frac{\varphi_t}{\varphi_{t-1}} < K_t^2$,

(ii) $1 > \frac{\varphi_t}{\varphi_{t-1}} > K_t^2$,

(iii) $1 = \frac{\varphi_t}{\varphi_{t-1}} = K_t$.

In particular, $\varphi_{t-1} < \varphi_t$ implies $\frac{\Gamma_{T-t}}{\Gamma_{T-(t-1)}} > \frac{\tau_v + \chi_t}{\tau_v + \chi_{t-1}}$.

Recall that $\chi_t \equiv \sum_{k=0}^{t} n^k \varphi_k$ measures a “stock” of information. Setting $\gamma \tau_v = 0$ shuts down a dynamic amplification mechanism. From (21), information aggregation in this case is self-defeating because $\frac{\tau_v + \chi_{t-1}}{\tau_v + \chi_t} < 1$. Hence, to accelerate info aggregation ($\varphi_{t-1} < \varphi_t$), there must be forward looking behavior $\gamma \tau_v > 0$ and trading must become more speculation-driven: $\frac{\Gamma_{T-t}}{\Gamma_{T-(t-1)}} > 1$. Because $\gamma \tau_v = 0$ makes $\frac{\Gamma_{T-t}}{\Gamma_{T-(t-1)}} = 1$, information aggregation cannot accelerate when either traders are myopic or have diffuse priors.
Lemma 4 (trading volume $E[|q_{it}|]$)

$\forall t = 1, \ldots, T,$

(a) $E[|q_{it}|] = \sqrt{\frac{2 \pi t}{\tau_{zt} x_t}}.$

(b) $\frac{E[|q_{it}|]}{E[|q_{it-1}|]} = \sqrt{\frac{\text{Var}[x_t]}{\text{Var}[x_{t-1}]}} \frac{x_{t-1}}{\sqrt{x_t x_{t-2}}}.$

Lemma 4(a) shows that the allocation dispersion and trading volume is positively related. Lemma 4(b) shows that the dynamics of trading volume depends on two factors: dynamics of allocation dispersion and the learning speed. Note that $\frac{x_{t-1}}{\sqrt{x_t x_{t-2}}} > 1 \iff \ln x_{t-1} > \frac{\ln x_{t-2} + \ln x_t}{2}.$ Hence, if the allocation dispersion increases and $\{x_{t-2}, x_{t-1}, x_t\}$ is increasing at the rate slower than the exponential rate in time (so that the log of them is concave), then the trading volume increases.

UNDER CONSTRUCTION.

Lemma 5 (volume and volatility)

Lemma 6 (information content of volume and prices)
3.3 Simulations

We numerically solve the model for $T = 10$, $n \in \{1, 2, 3\}$, and $\gamma \in \{0, 0.125, 0.25, 0.5\}$. The solutions show that $\tau_t$ increases in $\gamma$ while $\tau_{xt}$ decreases in $\gamma$ for given $n$ and $t$. Thus, as traders become more concerned with future trading rounds, more information is aggregated, while allocation becomes more dispersed. Importantly, the allocation dynamics can be quite non-monotonic as in Figure 1. In some cases, the dispersion can be higher than the initial dispersion at some point in the trading rounds. If the game ends at such time, traders learn about the tree from trading, but end up with the more dispersed tree allocation and the higher inventory cost.

4 Conclusion

This paper studied dynamic asset trading with two frictions: 1) trading is locally intermediated, and 2) all traders have private information. We analyzed how dispersed information is aggregated over time from intermediated trades. In the absence of private signals, the efficient allocation can be achieved after the sufficient number of trading rounds given that more than two traders are involved in each intermediated trading. With private signals, information gets aggregated over time, but the allocation may move away from the efficient allocation. When the degree of asymmetric information is high and traders put a larger probability weight on future trading rounds, a highly non-monotonic pattern in trading activity can arise.

5 Appendix 1. Under construction.

We solve the model for $T = 10$ and $\rho = 1$. Other model parameters are set as follows.

\[
\tau_{\varepsilon} = 0.05 \quad \tau_{x} = 5 \quad n \in \{1, 2, 3\}
\]

\[
\tau_{v} = 1 \quad \kappa = 2.5 \quad \gamma \in \{0, 0.125, 0.25, 0.5\}
\]
Appendix 2. Under construction.

(b) \( E [q_t^{xy}] = \frac{n}{n+1} E \left[ \frac{1}{n} \sum_{j \neq i} x_{j0} - x_{i0} \right] = \frac{n}{n+1} \sqrt{\frac{2}{n} V \left[ \frac{1}{n} \sum_{j \neq i} x_{j0} - x_{i0} \right]} = \sqrt{\frac{2}{n} \frac{n+1}{n} \tau_x} \).

\[ \frac{\tau_\varepsilon}{\tau_{xt}} = \frac{\rho \chi_{t-1} \chi_t}{\eta_t \varphi_t} \left( \frac{\Gamma_{T-t}}{\tau_\varepsilon \chi_t} \right)^2. \]

UNDER CONSTRUCTION.

We provide a series of lemmas that characterize the dynamic equilibrium.

Lemma 5 \((\Gamma_{T-t})\)

(a) For t = 1, .., T - 1, \( \Gamma_{T-t} \in \left[ \frac{\frac{\tau_x}{\tau_\varepsilon} + \chi_t + 1}{\frac{\tau_x}{\tau_\varepsilon} + (1-\gamma) \chi_{t+1}}, (1-\gamma) \frac{\tau_x}{\tau_\varepsilon} + \chi_{t+1} \right] \).

If \( \tau_x = 0 \), \( \Gamma_{T-t} = \frac{1}{1-\gamma} \) \( \forall t = 1, .., T - 1 \).

If \( \gamma = 0 \), \( \Gamma_{T-t} = 1 \) \( \forall t = 1, .., T \).

(b) Given \( \gamma \frac{\tau_x}{\tau_\varepsilon} > 0 \), \( \frac{\Gamma_{T-t}}{\Gamma_{T-(t-1)}} \in \left( 1, \frac{\frac{\tau_x}{\tau_\varepsilon} + \chi_t}{(1-\gamma) \frac{\tau_x}{\tau_\varepsilon} + \chi_t} \right) \) for \( t = 1, .., T - 1 \).

(c) As \( \gamma \to 1 \), \( \Gamma_{T-t} \to \infty \) for \( t = 1, .., T - 1 \).

(b) \( K_t = \frac{\frac{\tau_x}{\tau_\varepsilon} + \chi_{t-1}}{\frac{\tau_x}{\tau_\varepsilon} + (1+\gamma) \chi_{t-1}} \chi_t \Gamma_{T-t} \).

Lemma 7 (allocation convergence \( \frac{\text{Var}[x_t]}{\text{Var}[x_{t-1}]} \))

\( \forall t = 2, .., T, \)

(a) \( \frac{\text{Var}[x_t]}{\text{Var}[x_{t-1}]} = \frac{\chi_{t-1} + n^t \varphi_t}{n \chi_{t-1} - n^t \varphi_t} > \frac{1}{n} \).

(b) \( \frac{\text{Var}[x_t]}{\text{Var}[x_{t-1}]} \leq 1 \Leftrightarrow \varphi_t \in \left( 0, \frac{n-1}{2n} \frac{\chi_{t-1}}{n^t} \right] \) and \( \frac{\text{Var}[x_t]}{\text{Var}[x_{t-1}]} > 1 \Leftrightarrow \varphi_t \in \left( \frac{n-1}{2n} \frac{\chi_{t-1}}{n^t}, \frac{\chi_{t-1}}{n^t} \right) \).

In particular, \( \frac{\text{Var}[x_t]}{\text{Var}[x_{t-1}]} > 1 \) if \( n = 1 \).
References


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