Deferred-Acceptance Auctions and Radio
Spectrum Reallocation*

Paul Milgrom†        Ilya Segal‡

May 9, 2014

Abstract

Deferred-acceptance auctions choose allocations by an iterative process of rejecting the least attractive bid. These auctions have distinctive computational and incentive properties that make them suitable for application in some challenging environments, such as the planned US auction to repurchase television broadcast rights. For any set of values, any deferred acceptance auction with “threshold pricing” is weakly group strategy-proof, can be implemented using a clock auction, and leads to the same outcome as the complete-information Nash equilibrium of the corresponding paid-as-bid auction. A paid-as-bid auction with a non-bossy bid-selection rule is dominance solvable if and only if it is a deferred acceptance auction.

*The authors thank Alexey Kushnir as well as Mohammad Akbarpour, Piotr Dworczak, Tim Roughgarden, Inbal Talgam Cohen, Joshua Thurston-Milgrom, and Alex Wolitzky for their comments and suggestions.
†Department of Economics, Stanford University, Stanford, CA 94305. The authors are consultants to the US Federal Communications Commission concerning the design of the US “incentive auction.” This research analyzes some properties of proposals to which the authors have contributed.
‡Segal gratefully acknowledges the support of the Toulouse Network for Information Technology.
1 Introduction

We study a class of heuristic auctions that could be used for computationally challenging resource allocation problems. One classical example is combinatorial auctions. A more recent example – and the one that motivates this study – arises from the US government’s effort to reallocate channels currently allocated for television broadcasting to use for wireless broadband services. This reallocation involves purchasing television broadcast rights from some TV stations, reassigning the remaining over-the-air broadcasters into a smaller set of channels (known as “repacking”), using the cleared spectrum to create licenses suitable for use in wireless broadband, and selling those licenses to cover the costs of acquiring broadcast rights.\textsuperscript{1} The repacking of about 2,000 TV stations must be done in a way that satisfies about 130,000 interference constraints, which dictate which pairs of stations could not be assigned to the same or adjacent channels. Even determining whether a given set of broadcasters can be feasibly assigned to the available set of channels is a computationally challenging problem, equivalent to the NP-hard “graph coloring problem” (see Aardal et al. (2007)). The problem of selecting the set of stations to maximize the total broadcast value in the remaining TV spectrum is an even harder computational problem, which cannot be solved exactly in reasonable time using today’s state-of the art algorithms and hardware. This creates a challenge for designing a strategy-proof auction: for example, the approximation error in computing Vickrey prices would be quite large relative to the bids of individual bidders, potentially creating large incentives for misrepresenting preferences.\textsuperscript{2}

\textsuperscript{1}Middle Class Tax Relief and Job Creation Act of 2012, Pub. L. No. 112-96, §§ 6402, 6403, 125 Stat. 156 (2012). The legislation aims to make it possible to reallocate spectrum from a lower-valued use to a higher-valued one, generating billions in net revenues for the federal government.

\textsuperscript{2}The problem of designing computationally feasible economic mechanisms is studied in the field of “Algorithmic Mechanism Design” (Nisan and Ronen (1999)). While economists have long been concerned about the computational properties of economic allocation mechanisms (e.g., see Hayek (1945)), the formal economic literature has focused on model-
Our proposed solution is to use the class of “deferred-acceptance” (DA) auctions, which generalize the idea of the eponymous matching algorithms of Gale and Shapley (1962) and Kelso and Crawford (1982) and related clock auction designs, such as the simultaneous multiple-round auctions used for radio spectrum sales. These auctions are iterative processes in which the highest scoring (and hence least attractive) bids are rejected in each round and the bids left standing at the end are accepted.

For concreteness, we focus on the procurement setting (the modification to the selling setting is straightforward). We also restrict attention to settings in which each bidder is “single-minded,” meaning that it offers one good (or bundle of goods) for sale and cares only about winning or losing and the amount of its compensation. A DA auction in this setting is based on the following kind of DA algorithm: All the bids are initially active. In each iteration, each bid is assigned a non-negative “score,” which is based on the bid amount, the bidder’s identity, the remaining set of active bidders, and the amounts and identities of the inactive (previously rejected) bids. If no active bid has a positive score, the algorithm terminates and the remaining active bids become winning. Otherwise, the algorithm rejects the active bid with the highest non-zero score, removing it from the active set, and iterates.

Deferred-acceptance auctions are sealed-bid mechanisms that select winning bids using a DA algorithm and make payments only to winning bidders. In particular, we consider auctions that pay each winning bidder his threshold price, defined as the highest bid he could have made while still winning, holding all the other bids fixed. It is routine to show that any deferred-acceptance threshold auction, which uses a DA algorithm to select winners and pays each winner its threshold price, is a strategy-proof auction.

The class of DA algorithms and the auctions based on them is quite broad, and can be tailored to achieve a variety of different goals, nesting a number of
previously studied auctions. For the special case in which the goal is to find an efficient allocation subject to constraints that have a matroid structure (informally, when adding one bidder displaces at most one other bidder), the goal can be achieved exactly using the “standard” DA algorithm of Bikhchandani et al. (2011) in which all active bids are scored by their bid values. In settings with more complex constraints, DA algorithms may not be able to achieve exact efficiency, but may be able to approximate it fairly closely. For example, a DA auction can implement Dantzig’s greedy algorithm for the “knapsack problem” in which the rejected bids must satisfy a constraint on the total “volume.” For the FCC’s repacking problem, our simulations using appropriate scoring for stations based on their interference constraints have generated outcomes, for sample station values, that are within percentage points of exact efficiency.3

In many cases, efficiency is not the only or even the primary requirement. For example, the auctioneer may face a hard budget constraint (as the FCC does in its repacking problem). We show how scoring in DA threshold auctions may be designed to achieve this goal, nesting the budget-constrained mechanisms previously studied by McAfee (1992), Moulin (1999), Juarez (2007), Mehta et al. (2007), Singer (2010), and Ensthaler and Giebe (2014). Another important goal is to minimize the procurement cost, and if value distributions are known, then scoring can be used to incorporate Myerson’s (1991) virtual-values into DA auctions. Also, if bidders’ values are informative about each other (as in the case of i.i.d. draws from an unknown distribution), then expected costs can be reduced by conditioning scores on the values of already-rejected bids (similarly to the “yardstick competition” in Segal (2003)).

3The auction requires checking in every round whether any given active station could be feasibly rejected and added to the already-assigned stations with possible repacking. While this is an NP-hard computational problem, it turns out that this computation could be sped up to a matter of seconds. When the computation does time out, which happens occasionally, we treat the answer as infeasibility and so cannot reject the bidder.
The main theoretical contribution of the paper lies in establishing several nice properties that are satisfied by DA threshold auctions. First, we establish an equivalence between the sets of DA threshold auctions and descending “clock” auctions. Intuitively, similarly to a DA auction, a clock auction iteratively rejects the bids that are currently the least attractive to the auctioneer, but unlike the DA auction it permits the rejected bidders to remain active by improving their bids by some minimal bid increment. Formally, we show that for every sealed-bid DA threshold auction with finite bid spaces, there exists an equivalent descending clock auction that generates the same allocation and payments, and vice versa. Given the theoretical equivalence, clock auctions may be preferable for many practical applications, because they (i) make strategy-proofness self-evident even for bidders who misunderstand or mistrust the auctioneer’s calculations (see Kagel (1987) for experimental evidence), (ii) do not require winning bidders to reveal or even figure out their exact values, and (iii) permit information feedback so as to better aggregate common-value information.

A second nice property of DA threshold auctions and the corresponding clock auctions is weak group strategy-proofness: no coalition of bidders has a joint deviation from truthful bidding that is strictly profitable for all members of the coalition. For the clock auctions, this property holds regardless of what information is disclosed to bidders during the auction. This result extends weak group strategy-proofness results obtained earlier by Moulin (1999), Mehta et al. (2007), and Hatfield and Kojima (2009).

The third set of results concerns properties of paid-as-bid auctions in which winners are selected using a DA algorithm. We show that in the full-information setting, a paid-as-bid DA auction has a Nash equilibrium that generate the same outcome (the set of winners and payments to those winners) as the threshold auction using the same DA algorithm. While the assumption of full information is unrealistic for many applications, it nevertheless offers a tractable benchmark to compare the performance of the two
pricing rules and conclude that threshold prices do not necessarily raise the auctioneer’s cost. Furthermore, when we limit attention to “non-bossy” DA algorithms (in which a bidder cannot alter any part of the allocation without changing his own allocation) and to generic values, the outcome of the threshold auction is also the unique undominated Nash equilibrium outcome of the paid-as bid auction, and also the unique outcome surviving iterated elimination of weakly dominated strategies. This property is not shared by any other non-bossy bid selection rule: any non-bossy paid-as-bid auction that is generically dominance solvable is a DA auction.

To highlight how special are the properties of DA auctions, we check the properties of other auctions with single-minded bidders in which winning bids are selected in other ways. Two other kinds of bid selection rules have been proposed. First, some auctions are based on optimization, in which the collection of winning bids is the one that minimizes the total bid. When the constraints have a matroid structure that ensures that bidders are substitutes, these auctions can be implemented as DA auctions and therefore have all the nice properties described above, but otherwise they do not: they (i) cannot be implemented as clock auctions, (ii) are not weakly group strategy-proof, and (iii) yield threshold prices (Vickrey prices) that result in a higher procurement cost than the full-information Nash equilibria of the corresponding paid-as-bid auction (see Bernheim and Whinston (1986)). Second, Lehmann et al. (2002) have introduced auctions that select winners using a “greedy-acceptance” algorithm, in which bids are accepted iteratively until certain

\footnote{Ausubel and Milgrom (2006) propose a “cumulative-offer” clock auction which implements efficiency, but this auction requires the ability to “recall” exited bids, and this violates strategy-proofness.}

\footnote{Indeed, this high costs/low revenue problem of the Vickrey auction is the motivation for the proposed “core-selecting auctions” (Day and Milgrom 2008), which sacrifice strategy-proofness in order to increase revenue. The present paper proposes a different solution to the problem, preserving strategy-proofness but sacrificing the efficiency of outcome according to the bids. (Of course, core-selecting auctions only guarantee that the outcome is efficient according to the bids, and so efficiency of outcome according to true values is not ensured given the gaming opportunities generated by these auctions.)}
In general, these greedy-acceptance auctions permit the calculation of threshold prices that ensure strategy-proofness, and they can achieve good approximations of efficiency in some interesting, computationally challenging applications. (Furthermore, when the constraints have a matroid structure that ensures that bidders are substitutes, there exists a greedy-acceptance algorithm that yields the optimizing allocation rule and the threshold prices equal to the Vickrey prices, just as it is the case for the corresponding deferred-acceptance auction.) However, we show by means of an example that in general greedy-acceptance algorithms have none of the other nice properties described above: they (i) cannot be implemented as clock auctions, (ii) are not weakly group strategy-proof, and (iii) can result in a higher procurement cost than the full-information Nash equilibria of the corresponding paid-as-bid auction.

The paper is organized as follows: Section 2 introduces our model and characterizes strategy-proof sealed-bid auctions in our setting. Section 3 introduces deferred-acceptance auctions. Section 4 provides several examples of useful DA auctions. Section 5 introduces descending-clock auctions in which winners are paid their final clock prices and shows the equivalence between these auctions and DA threshold auctions. Section 6 shows that descending clock auctions (and the equivalent threshold-price DA auctions) are weakly group strategy-proof. Section 7 analyzes paid-as-bid DA auctions, deriving their properties described above. Section 8 concludes with a discussion of how some of these results might be used for the FCC’s auction design problem. In an appendix, we show by a series of examples that the properties of DA auctions are not shared by auctions based on optimization or by auctions based on greedy acceptance algorithms.

For other examples of such auctions, see also Mu’alem and Nisan (2008), Babaioff and Blumrosen (2008), and the references therein.
2 Sealed-Bid Auctions

Let $N$ be the set of bidders. In the auction, each bidder either “wins” (which means that his bid to supply a given good or set of goods is “accepted”) or “loses” (which means that his bid is rejected). We restrict attention to mechanisms, which we call “auctions,” in which winning bidders receive payments but losing bidders do not.

The preferences of each bidder $i$ depend on whether he wins or loses, and, if he wins, on the payment $p_i$. We assume that these preferences are strictly increasing in the payment and that there exists some payment $v_i$ that makes him indifferent between winning and losing; we call $v_i$ his “value”. The set of bidder $i$’s possible values is $[0, v_i]$ for each $i$.

We restrict each bidder to make bids in a finite set $B_i \subseteq \mathbb{R}_+$ and assume that $\max B_i > v_i$. This ensures that a non-participation strategy is never better for bidder $i$ than bidding $\max B_i$, so we can simplify our formulation by not allowing non-participation.

A sealed-bid auction takes a bid $b_i \in B_i$ from each bidder $i = 1, \ldots, N$. For every bid profile $b \in B = \Pi_i B_i$, the auction’s allocation rule $\alpha : B \to 2^N$ determines a set of winning bids $\alpha (b) \subseteq N$ and its payment rule $p : B \to \mathbb{R}^N$ specifies a corresponding payment profile, with $p_i (b) = 0$ for $i \in N \setminus \alpha (b)$ (so losing bidders are not paid). A sealed-bid auction is a triple $\langle B, \alpha, p \rangle$.

With finite bid sets, it is convenient to replace the usual notion of “truthful” bidding by a similar concept of “strategy-proofness” that applies even when truthful reporting is infeasible because $v_i \notin B_i$. According to our definition, an auction is strategy-proof if it is always optimal for a bidder to round up its value to the next allowable bid.

**Definition 1** The sealed-bid auction $\langle B, \alpha, p \rangle$ is strategy-proof if for every bidder $i$, $v_i \in [0, \bar{v}_i]$, and $b_{-i} \in B_{-i}$, it is optimal for bidder $i$ to bid $v_i^+ \equiv \min \{b_i \in B_i : b_i > v_i\}$.

---

7For unmixed outcomes, such a preference can be expressed by a quasilinear utility $p_i - v_i$ when the bidder wins and zero when he loses.
The usual characterization of strategy-proof auctions applies in this setting. For completeness, we restate it here.

**Definition 2**  The allocation rule $\alpha$ is monotonic if and only if $i \in \alpha (b_i, b_{-i})$ and $b_i' < b_i$ imply $i \in \alpha (b_i', b_{-i})$.

**Definition 3**  A sealed-bid auction $(B, \alpha, p)$ is a threshold auction if and only if $\alpha$ is monotonic and the price paid to any winning bidder $i \in \alpha(b)$ satisfies the threshold pricing formula:

$$p_i(b_{-i}) = \max \{ b_i' \in B_i : i \in \alpha (b_i', b_{-i}) \}.$$  

(1)

With these definitions, a standard argument implies the following result.

**Proposition 1**  A sealed-bid auction $(B, \alpha, p)$ is strategy-proof if and only if it is a threshold auction.\textsuperscript{8}

### 3  Deferred-Acceptance Auctions

Given a set of bidders $N$ and possible bid profiles $B$, a deferred-acceptance algorithm (DA) is specified by a collection of scoring functions $(s^A_i)_{A \subseteq N, i \in A}$, where for each $A \subseteq N$ and each $i \in A$, the function $s^A_i : B_i \times B_{N \setminus A} \rightarrow \mathbb{R}_+$ is nondecreasing in its first argument. The algorithm operates as follows. Let $A_t \subseteq N$ denote the set of “active bidders” at the beginning of iteration $t$. We initialize the algorithm with $A_1 = N$. In each iteration $t \geq 1$, if $s^A_t (b_i, b_{N \setminus A_i}) = 0$ for all $i \in A_t$ then stop and output $\alpha(b) = A_t$; otherwise, let $A_{t+1} = A_t \setminus \arg \max_{i \in A_t} s^A_t (b_i, b_{N \setminus A_i})$ and continue. In words, the algorithm iteratively rejects the least desirable (highest-scoring) bids until only

\textsuperscript{8}It is obvious that a threshold auction is strategy-proof. For the converse, apply the envelope theorem to see that the outcome function and strategy-proofness determine the bidders’ payoffs up to a constant of integration determined. Since losers must be paid zero and have a payoff of zero, the strategy-proof payment function is unique.
zero scores remain. We say that the DA algorithm computes allocation rule \( \alpha \) if for every bid profile \( b \in B \), when the algorithm stops the set of active bidders is exactly \( \alpha (b) \).

By inspection, every DA algorithm is finite and computes a monotonic allocation rule. Thus, we can define a DA threshold auction as a sealed-bid auction which computes its allocation using a DA algorithm and makes the corresponding threshold payments \( (1) \) to the winners. Like any threshold auction, each DA threshold auction is strategy-proof. Furthermore, the threshold prices can be computed in the course of the DA algorithm, by initializing the prices as \( p^0_i = \sup B_i \) for all \( i \), and then updating them in each round \( t \geq 1 \) as follows:

\[
p^t_i(b) = \min \{ p^{t-1}_i, \sup \{ b'_i \in B_i : s^A_i (b'_i, b_{N \backslash A_t}) < s^A_j (b_j, b_{N \backslash A_t}) \text{ for } j \in A_t \backslash A_{t+1} \} \}
\]

for every bidder \( i \in A_{t+1} \). In the final round of the algorithm, for every winner \( i \in A_T \), \( p^T_i(b) \) is the winner’s threshold price.

4 Examples of Deferred-Acceptance Auctions

Example 1 (Feasibility and Non-Wastefulness) Suppose that the procurement goal is to acquire products from a subset of bidders \( A \in \mathcal{F} \), where \( \mathcal{F} \subseteq 2^N \) is exogenously given. We assume that \( N \in \mathcal{F} \) so that the procurement goal is achievable). To ensure that the algorithm maintains feasibility, we require that \( s^A_i (b_i, b_{N \backslash A}) > 0 \) only if \( A \backslash \{i\} \in \mathcal{F} \), and that there are no ties, i.e., \( s^A_i (b_i, b_{N \backslash A}) \neq s^j_j (b_j, b_{N \backslash A}) \) for all \( i \neq j \), \( A, b_i, b_j, b_{N \backslash A} \).

Say that the family \( \mathcal{F} \) is comprehensive if \( A \in \mathcal{F} \) and \( A \subseteq A' \) implies \( A' \in \mathcal{F} \) (i.e., it has “free disposal” of winners). Say that the algorithm has perfect feasibility checking if \( s^A_i (b_i, b_{N \backslash A}) > 0 \) if and only if \( A \backslash \{i\} \in \mathcal{F} \) – that is, it stops only when it is infeasible to reject any more active bids. If both conditions hold, then the algorithm has the desirable property of non-wastefulness – i.e., it stops at a minimal set \( A \in \mathcal{F} \). (Indeed, if there is
some $A' \in \mathcal{F}$ that is a strict subset of $A$, then for each $i \in A \setminus A'$ we have $A \setminus \{i\} \in \mathcal{F}$ by comprehensiveness, and therefore $s_i^A (b_i, b_{N \setminus A}) > 0$ by perfect feasibility checking, hence the algorithm cannot stop at $A$.)

In some settings, however, perfect feasibility checking may be too computationally challenging, and imperfect checking must be used instead. For example, in the FCC’s spectrum-clearing problem, a given set $A$ of bidders is in $\mathcal{F}$ when there exists an assignment of the rejected bidders $N \setminus A$ to available channels that satisfies all interference constraints (as formalized in (3) below), so checking that requires solving an NP-hard problem. When a feasibility checker has a limited time to run on a computationally challenging problem, its output may report (i) that $A \setminus \{i\} \in \mathcal{F}$ (confirmed by generating a feasible assignment of the rejected bidders to channels), (ii) that $A \setminus \{i\} \notin \mathcal{F}$ (confirmed by a “proof” that such a feasible assignment does not exist), or (iii) time-out, which means that the algorithm must terminate before generating either (i) or (ii). In case (i), we set $s_i^A (b_i, b_{N \setminus A}) > 0$, while in cases (ii) and (iii), we need to set $s_i^A (b_i, b_{N \setminus A}) = 0$ in order to guarantee that the algorithm yields a feasible assignment. Without perfect feasibility checking, the outcome may not be non-wasteful.

**Example 2 (Standard DA algorithm for Optimization with Matroid Constraints)**

Suppose that the goal is to find an “efficient” (i.e. social cost-minimizing) set of winning bids subject to the feasibility constraint in Example 1:

$$\alpha (b) \in \arg \min_{\sum_{i \in A} b_i} \quad \text{(2)}$$

Consider the “standard” DA algorithm that does perfect feasibility checking and scores the bids that are feasible to reject by their bid amounts, i.e.,

$$s_i^A (b_i, b_{N \setminus A}) = \begin{cases} b_i & \text{when } A \setminus \{i\} \in \mathcal{F}, \\ 0 & \text{otherwise,} \end{cases}$$

and assume that there are no ties (i.e., $b_i \neq b_j$ for $i \neq j$ for each $b \in B$). By a classical result in matroid theory (see Oxley (1992)), the standard DA algorithm computes an efficient allocation rule $\alpha$ in this setting if and only if the set family $\{R \subseteq N : N \setminus R \in \mathcal{F}\}$ –
i.e., the feasible sets of rejected bids – form the independent sets of a matroid on the ground set $N$.\(^9\)

Intuitively, the matroid property in Example 2 involves a notion of “one-for-one substitution” between bidders. (In particular, it implies that all the maximal feasible sets of rejected bids – the “bases” of the matroid – have the same cardinality.) This does not hold, even approximately, in the FCC setting, which involves a trade-off between acquiring a larger number of stations with smaller coverage areas and a smaller number of stations with larger coverage areas. In such cases, instead of the standard DA algorithm it is desirable to use scoring that assigns greater scores to larger stations, given the same bid values. This is illustrated in the following stylized example:

**Example 3 (Scoring for the Knapsack Problem)** Suppose that the family of feasible sets takes the form

$$
\mathcal{F} = \left\{ A \subseteq N : \sum_{i \in N \setminus A} w_i \leq 1 \right\}.
$$

The problem of maximizing total surplus subject to this constraint is known as the “knapsack problem.” Interpreting $w_i > 0$ as the “volume” of bidder $i$, the problem (2) is to minimize the total value of the accepted bids subject

\(^9\)This is closely related to the analysis of Bikchandani et al. (2011), who focus on “selling” auctions in which the family of sets of bids that could be feasibly accepted is a matroid. For these settings, the DA heuristic achieving efficiency is the “greedy worst-out” heuristic: reject the lowest bid that could be rejected while the set of accepted bids still forming a spanning set of a matroid. (Note that in contrast to the “standard” DA procurement auction, the provisional allocation in the selling auction remains infeasible until the end of the auction.) By the argument in Section 5 below, this DA heuristic corresponds to the ascending-price clock auction constructed by Bikchandani et al. (2011) in which all the active bidders that could still be rejected while maintaining the spanning property always face the same price. While both the “greedy best-in” and the “greedy worst-out” algorithms can be used to obtain compute the efficient allocation rule in both the procurement and selling matroid auctions, only one of them is a DA algorithm: it is the best-in algorithm for the rejected bids in the procurement auction, and the worst-out algorithm for the rejected bids in the selling auction.
to a constraint on their total volume.\footnote{Duetting, Gkatzelis and Roughgarden (2014) consider instead the “selling” problem in which the accepted bids must fit into a knapsack, and examine the approximation power of DA algorithms for this problem.} While this optimization problem is NP-hard, it can be approximated with the famous heuristic of Dantzig (1957), which corresponds to the DA algorithm with the “per-volume” scoring
\[
s_t^A(b_i, b_{N\setminus A}) = \begin{cases} 
  b_i/w_i & \text{when } A\setminus\{i\} \in \mathcal{F}, \\
  0 & \text{otherwise}.
\end{cases}
\]

In the FCC setting, keeping total procurement cost low is also an important goal. A cost-minimization objective can be easily handled using the “virtual value” approach of Myerson (1981).

**Example 4 (Scoring for Expected Cost Minimization)** Suppose that bidders’ values $v_i$ are independently drawn from distributions $F_i(v) = \Pr\{v_i \leq v\}$, and that $B_i = [0, \bar{b}_i]$ for each $i$ (so that bidding exact values is possible). Then, following the logic of Myerson (1981), the expected cost of a threshold auction that implements allocation rule $\alpha$ can be expressed as
\[
\mathbb{E}\left[\sum_{i \in \alpha(v)} \gamma_i(v_i)\right], \quad \text{where } \gamma_i(v_i) = (v_i + F_i(v_i)/F'_i(v_i)) \text{ (bidder } i \text{'s “virtual cost function”).}
\]
Assume the virtual cost functions are strictly increasing. Then, if we are given a DA algorithm with scoring rule $s$ that exactly or approximately minimizes the expected social cost subject to feasibility constraints as in the above examples, it can be modified to yield a DA threshold auction that exactly or approximately minimizes the total expected cost of procurement subject to the constraints. The modified auction uses the scoring rule
\[
\tilde{s}_t^A(b_i, b_{N\setminus A}) = s_t^A(\gamma_i(b_i), (\gamma_j(b_j))_{j \in N\setminus A}).
\]

The previous examples have used “fixed” scoring, which conditions only on the bidder’s identity and his bid value. The following two examples demonstrate two reasons why it may be useful to use “adaptive” scoring, that is, to condition on the set of rejected bids and their bid values.
Example 5 (Adaptive Scoring for a Budget Constraint) Suppose that the designer faces the “budget constraint” that the total payment to the winners cannot exceed $R(A)$ when the set of accepted bids is $A \subseteq N$. In order to find a maximal set of winners satisfying the budget constraint, the algorithm needs to continue whenever the total threshold prices to be paid to the winners exceeds $R(A)$. If the budget constraint is the only limit on procurement, then the algorithm stops as soon as that limit is satisfied. For a simple example, if scoring were based on increasing functions $\sigma_i : B_i \rightarrow \mathbb{R}_{++}$ (depending only on the $i$’s bid), then the threshold price that would have to be paid to any currently active bidder $i \in A$ if the algorithm were to stop right away is:

$$p_i (b_{N\setminus A}^i) = \sup \left\{ b_i^j \in B_i : \sigma_i (b_i^j) < \max_{j \in N\setminus A} \sigma_j (b_j) \right\}.$$

In this case, the “stopping rule” described above can be implemented via the scoring functions

$$s_i^A (b_i, b_{N\setminus A}^i) = \begin{cases} 
\sigma_i (b_i) & \text{if } \sum_{j \in A} p_j (b_{N\setminus A}^j) > R (A), \\
0 & \text{otherwise}.
\end{cases}$$

These kinds of algorithms may be called “revenue-sharing” algorithms, and their mirror images for the “selling” auction are the “cost-sharing” algorithms of Moulin (1999) and Mehta et al. (2009). (While this formulation permits the auction to generate a positive net revenue, it is possible to modify it to require exact budget balance.) An equivalent clock auction has been considered by Ensthaler and Giebe (2014) for the case where $R(A)$ is a constant.

It is also possible to accommodate both budget constraints and other feasibility constraints. For example, consider McAfee’s (1992) double-auction algorithm for a homogeneous-good market with unit buyers and unit sellers, in which the feasibility constraint dictates that the number of the accepted buying bids (“demand”) equal the number of the accepted selling bids (“supply”). McAfee’s algorithm can be viewed as a DA threshold auction that
rejects either the highest-cost selling bid or the lowest-value buying bid in each round, making sure that demand is always be within 1 of supply, and that stops as soon as the twin conditions are met: (i) demand equals supply and (ii) the net threshold payment is nonnegative. (This mechanism is equivalent to a clock auction in which all buyers face the same ascending price and all sellers face the same descending price.) The FCC’s “incentive auction” is similarly a double auction for spectrum sellers (TV broadcasters) and spectrum buyers (mobile broadband companies) that is constrained to generate a certain amount of net revenue, but subject to the added complication that buyers demand and sellers supply different kinds of differentiated goods and the feasible combinations of accepted bids are quite complicated. However, FCC’s problem also admits a DA double auction that is similar to McAfee’s double auction, which sets a sequence of $n$ possible spectrum targets for the number of channels to clear, and starts by trying to clear the largest number, reducing the target whenever the revenue goals cannot be achieved with the current spectrum clearing target.\footnote{Duetting, Roughgarden, and Talgam-Cohen (2014) consider the approximation power of balanced-budget DA double auctions for settings in which buyers and sellers must be matched one-to-one subject to some constraints.}

**Example 6 (Adaptive Scoring for Yardstick Competition)** Suppose that the auctioneer cares about the expected total profit, that her gross profit for acquiring each bidder is $\pi$, and that there are no other constraints on the feasible allocations. Further suppose that bidders’ values are random, i.i.d. draws from a distribution that is unknown to the auctioneer. In this symmetric setting we can focus without loss on auctions that treat bidders symmetrically. The simplest DA algorithms simply set a reserve price $p^*$ and buy from all bidders who quote a lower price. This can be implemented as a DA auction with $s_i^A(b_i) = \max\{v_i - p^*, 0\}$ to yields an expected profit on each bidder $i$ of $(\pi - p^*) \Pr\{v_i \leq p^*\}$, where the probability is calculated based on the auctioneer’s prior. However, it is evident that the auctioneer can do better.
in expectation by conditioning the price quoted to each still-active bidder on the information implied by the previously rejected bids. Letting \( p^*_A(b_{N\setminus A}) = \arg\max_p (\pi - p) \Pr \{ v_i \leq p|b_{N\setminus A} \} \) be the optimal monopsony price for the posterior distribution of values given the rejected bids, the better DA auction would set scores according to \( s^A_i(b_i, b_{N\setminus A}) = \max \{ v_i - p^*_A(b_{N\setminus A}) , 0 \} \).\(^{12}\)

4.1 Example: Near-Optimality of DA Algorithms

For computationally challenging problems in which full optimization is impossible, one may ask where there exists some DA algorithm achieve reasonably good performance. Partly, this question should be addressed by simulations. In this subsection, we examine a setting in which this question can be studied theoretically and which is similar to the FCC repacking problem.

In this setting, the bidders are television stations who bid to relinquish their broadcast rights. Broadcast channels must be assigned to the stations whose bids are rejected in a way that satisfies non-interference constraints. In this simplified model, those constraints are represented by an interference graph \( Z \): a set of two-elements subsets of \( 2^N \) (“edges”). We interpret \( \{i, j\} \in Z \) to mean that stations \( i \) and \( j \) cannot both be assigned to the same channel without causing unacceptable interference. Letting \( \{1, ..., n\} \) denote the set of channels left after the auction. Then, the feasible sets of accepted bids for the FCC’s repacking problem can be written as

\[
\mathcal{F} = \{ A \subseteq N : (\exists c : N\setminus A \rightarrow \{1, ..., n\}) (\forall s, s' \in N\setminus A)(\{i, j\} \in Z \Rightarrow c(s) \neq c(s')) \}.
\]

\(^{12}\)Auctions that are even more profitable are possible. The expected profit-maximizing threshold auction examined in Segal (2003) implements the allocation rule \(\alpha(b) = \{ i \in N : b_i \leq p^*(b_{\_\_i}) \} \), that is, it offers each bidder the optimal price given the information inferred from all the other bids. However, this allocation rule is typically not implementable with a DA algorithm, because in it the winning bidders could be “bossy” – affect the allocation while still winning. In contrast, in a DA algorithm winning bidders are always non-bossy.
Proposition 2 Suppose there exists an ordered partition of the set $N$ of stations into $m$ disjoint sets $N_1, \ldots, N_m$ such that

(i) for each $k = 1, \ldots, m$ and each $i, j \in N_k$, $\{i, j\} \in Z$ (that is, each $N_k$ is a “clique”),

(ii) there exists some $d < n$ such that for each $k = 1, \ldots, m$ and each $S \subseteq N_k$ satisfying $|S| \leq n - d$, we have

$$|S| + |\cap_{s \in S} \cup_{j < k} \{s' \in N_j : \{s, s'\} \in Z\}| \leq n.$$

Then there exists a DA algorithm that, for any set of bids, first rejects the most valuable stations in each partition element $N_k$, iterates in that way as long as that is feasible, and then continues in any way. For every such algorithm, the value of stations assigned is at least a fraction $1 - d/n$ of the optimal value.

Intuitively, the Proposition (proven in the Appendix) applies to the setting in which the stations can be partitioned into $m$ “metropolitan areas” in such a way that (i) no two television stations in the same area can be assigned to the same channel and (ii) the cross-area constraints are limited in the sense that if we have a set $S$ of no more than $n - d$ stations in one metropolitan area, there are no more than $n - |S|$ stations in lower-indexed areas that have interference constraints with all the stations in $S$.\(^\text{13}\) Using an argument based on Hall’s Marriage Theorem, condition (ii) ensures that it is possible to select any arbitrary $n - d$ stations in each area independently of each other and still be able to find a feasible assignment of these stations to channels. Since the optimal value is bounded above by assigning the $n$ most valuable stations in each area (which would be feasible if there were no inter-area constraints), the worst-case fraction of efficiency loss is bounded above by $(n - d)/n$.

\(^{13}\)While we assume that the partition is totally ordered for notational simplicity, the proposition can also be extended to cases in which partition elements form an ordered acyclic graph, by interpreting $<$ and $\leq$ as referring to a precedence relation.
We make several observations.

1. It is possible to satisfy condition (ii) while having several times more inter-area constraints than within-area constraints. To illustrate, suppose that all stations are arranged from north to south on a line and that each station interferes with its \(n - 1\) closest neighbors to the north as well as its \(n - 1\) closest neighbors to the south. Suppose that each successive group of \(n\) stations is described as a metropolitan area. Then, it is possible to assign all stations \((d = 0)\) to channels without creating interference just by rotating through the \(n\) channel numbers. In this example, there are just \(x = n(n - 1)/2\) constraints among stations within an area but \(2x\) constraints between those stations and ones in the next lower indexed area and another \(2x\) constraints involving stations in the next higher indexed area.

2. In general, there may be many ways to partition stations into cliques, and many ways to order any given partition. The Proposition formally applies to each partition, but the number \(d\) and therefore the approximation guarantee will vary depending on which partition is selected and how it is ordered.

3. The standard DA algorithm discussed in Example 2 is not among the ones described in the Proposition, and does not achieve the same performance guarantee. For example, suppose that there is some central area – area 1 – such that the stations in any area \(m\) are linked to all other stations in that area and to the stations area 1, but not to any other stations. Suppose that there are 2 channels available. Then the DA algorithms described in the proposition assign the single most valuable station in each area, thus assigning at least \(M\) stations in all and achieving at least half of the optimal value. If the two most valuable stations are in area 1, then the standard DA algorithm assigns just
those two channels, and none others. In worst case, that achieves just $1/(M - 1)$ of the optimal value.

4. The worst-case bound applies over all possible station values and incorporates both a conservatively high estimate of the optimum and a conservatively low estimate of the DA algorithm performance. To approach the worst-case bound, the optimum must be similar to assigning the $n$ most valuable stations in each area, and the DA algorithm must be unable to assign any other stations after the $n - d$ most valuable stations have been assigned in each area.

5. This example shows how it may be possible to design a DA algorithm — not the standard one — to take advantage of known properties of the feasible set. In applications like the FCC auction, since the interference graph is known before the auction, it will often be possible to apply a variety of analytical tools and simulations to find a DA algorithm that performs much better than the standard one.

5 Clock Auctions

Informally, a ("descending") clock auction is a dynamic mechanism that presents a declining sequence of prices to each bidder, with each presentation followed by a decision period in which any bidder whose price has been strictly reduced decides whether to exit or continue. Bidders that have never exited are called “active”; others are called “inactive.” Bidders who remain active when their prices are reduced are said, informally, to “accept” the lower price. When the auction ends, the active bidders become the winners and are paid their last (lowest) accepted prices. Different clock auctions are distinguished by their pricing functions, which determine the sequence of prices presented.

To formalize the intuitive description, we restrict attention to finite clock
auctions with discrete periods.\footnote{We say that the clock auction is \textit{finite} if there exists some } \textit{T} \text{ such the auction always stops by period} \textit{T}. The restriction to finite auctions and finite bid spaces avoids familiar technical difficulties, such as those associated with describing continuous time auctions and with defining dominance solvability for infinite games.\footnote{In a variant of the auction, all active bidders } \textit{i} \text{ in } \textit{A}_t \text{ may choose whether to exit. Although the results below are the same for the auction in the main text and this variant, the version in the main text is preferred because, when there is a feasibility constraint as in 1, it ensures that the clock auction always yields a feasible outcome.}

The active bidders in period \textit{t} are denoted by \( A_t \subseteq 2^N \). A period-\textit{t} history consists of the sets of active bidders in all periods up to period \textit{t}: \( A^t = (A_1, \ldots, A_t) \) such that \( A_1 \subseteq \ldots \subseteq A_1 = N \). Let \( H \) denote the set of all such histories. A descending clock auction is a price mapping \( p : H \to \mathbb{R}^N \) such that for all \( t \geq 2 \) and all \( A^t \), \( p(A^t) \leq p(A^{t-1}) \). (Note that we reuse the \( p \) notation here to represent the pricing in the clock auction: it had earlier referred to pricing in the threshold auction.)

The clock auction initializes \( A_1 = N \). In each period \( t \geq 1 \), given history \( A^t \), a profile of prices \( p(A^t) \) is offered to the bidders. If \( p(A^t) = p(A^{t-1}) \), the auction stops; bidder \( i \) is a winner if and only if \( i \in A_t \) and in that case \( i \) is paid \( p_i(A^t) \). If \( i \in A_t \) and \( p_i(A^t) < p_i(A^{t-1}) \), then \( i \) may choose to refuse the new price and exit the set of active bidders.\footnote{In a variant of the auction, all active bidders } \textit{i} \text{ in } \textit{A}_t \text{ may choose whether to exit. Although the results below are the same for the auction in the main text and this variant, the version in the main text is preferred because, when there is a feasibility constraint as in 1, it ensures that the clock auction always yields a feasible outcome.} Letting \( E_t \subseteq A_t \) denote the set of bidders who choose to exit, the auction continues in period \( t + 1 \) with the new set of active bidders \( A_{t+1} = A_t \setminus E_t \) and the new history \( A^{t+1} = (A^t, A_{t+1}) \).

To complete the description of the auction as an extensive-form mechanism, we also need to describe bidders’ information sets. We allow general information disclosure: bidder \( i \) observes some signal \( \sigma_i(A^t) \) in addition to his current price \( p_i(A^t) \) in history \( A^t \).

A strategy for bidder \( i \) in a clock auction is a \textit{cutoff strategy} with cutoff \( b_i \) if it specifies that the bidder should exit if and only if \( p_i(A^t) < b_i \), for some \( b_i \leq p_i(N) \). Note in particular that every cutoff strategy accepts the opening price. The next two results show that clock auctions in which bidders are restricted to cutoff strategies (for example, in which they must use
proxy bidders with cutoff strategies) are equivalent to DA threshold auctions, meaning that the mapping from the profile of bids or cutoffs to allocations and prices are the same for both auctions.\footnote{If we were to implement a deferred-acceptance threshold auction as a multi-round procedure in which some information is disclosed to active bidders between rounds and the active bidders are allowed to improve their bids, then the resulting mechanisms would be the “survival auctions” proposed by Fujishima et al. (1999). These auctions are strategically equivalent to clock auctions without the restriction to cutoff strategies.}

**Proposition 3** For every DA threshold auction, there exists an equivalent finite clock auction in which bidders are restricted to cutoff strategies.

**Proof.** Given bid spaces $B_1, \ldots, B_N$, for each $v \in \mathbb{R}$, let and $v^+ = \min \{ b_i \in B_i : b_i > v \}$ and $v^- = \begin{cases} \max \{ b_i \in B_i : b_i < v \} & \text{if } v > \min B_i; \\ \min B_i - 1 & \text{otherwise.} \end{cases}$ Let the opening prices be $p_i(N) = \max B_i$ for each $i$. Given a DA threshold auction with scoring rule $s$, we construct an equivalent clock auction as follows: The price reduction rule in the clock auction reduces the price to every highest-scoring active bidder by the minimal amount, while leaving prices unchanged for the other bidders:

$$p_i(A^t) = \begin{cases} p_i(A^{t-1})^{-} & \text{if } i \in \arg \max_{j \in A_t} s_j^{A_t} (p_j(A^{t-1}), p_{N \setminus A_t}(A^t)^+) \\ p_i(A^{t-1}) & \text{otherwise.} \end{cases}$$

Note in particular that the auction maintains $p_i(A^t) = p_i(A^{t-1})$ for all $i \in N \setminus A_t$ – thus memorizing the prices rejected by bidders who have quit, so that their cutoffs can be inferred as $p_i(A^t)^+$.

Then equivalence is easy to see: First, for every history of the clock auction, the next set of bidders to quit in the clock auction is the set of bidders who have the maximum scores among the set of active bidders, so the set of winners is the same in both auctions. Second, if any winning bidder had said “no” to any higher price, it would have exited, so each bidder’s final clock price is the highest cutoff it could use to be winning – its threshold price.
Proposition 4 For every finite clock auction in which bidders are restricted to cutoff strategies, there exists an equivalent DA threshold auction.

Proof. Given a finite clock auction \( p \), we construct bid spaces and a scoring rule to create an equivalent DA threshold auction. We take each bidder \( i \)'s bid space to be \( B_i = \{ p_i(h) : h \in H \} \) – the set of possible prices agent \( i \) could face in the clock auction (which is a finite set in a finite clock auction).

Next, we construct the scoring rule in the following manner: Holding fixed a set of bidders \( S \subseteq N \) and their bids \( b_S \subseteq \mathbb{R}^S \), let \( A_t(S, b_S) \) denote the set of active bidders in the clock auction at round \( t \) in which every bidder \( j \in S \) uses cutoff strategy \( b_j \) and every bidder from \( N \setminus S \) never exits. Formally, initialize \( A_1(S, b_S) = N \) and iterate by setting

\[
A_{t+1}(S, b_S) = A_t(S, b_S) \setminus \{ j \in S : b_j > p_j \left( A^t(S, b_S) \right) \}.
\]

This gives an infinite sequence \( \{ A_t(S, b_S) \}_{t=1}^{\infty} \), but the sets start repeating at some point (when the clock auction stops).

Now for given \( A, b_{N \setminus A}, i \in A \), and \( b_i \), define the score of agent \( i \) as the inverse of how long he would remain active in clock auction if he uses cutoff \( b_i \) and all bidders from \( N \setminus A \) use cutoffs \( b_{N \setminus A} \), while bidders in \( A \setminus \{ i \} \) never quit:

\[
s_i^A(b_i, b_{N \setminus A}) = 1/\sup \{ t \geq 1 : i \in A_t \left( \{ i \} \cup (N \setminus A), (b_i, b_{N \setminus A}) \right) \}.
\]

(Note that the score is \( 1/\infty = 0 \) in cases in which the auction stops with agent \( i \) still active.) This score is by construction nondecreasing in \( b_i \). Also by construction, given a set \( A \) of active bidders, the set of bidders to be rejected by the algorithm in the next round \( (\arg \max_{i \in A} s_i^A(b_i, b_{N \setminus A})) \) is the set of bidders who would quit the soonest in the clock auction given that the inactive bidders have used cutoffs \( b_{N \setminus A} \). If no more bidders would exit the auction, then all active bidders have the score of zero, so the algorithm
stops. Finally, as argued above, the winners’ clock auction prices are their threshold prices: the winner would have lost by using any higher cutoff in $B_i$ than its clock auction price.

6 Group Strategy-proofness

**Definition 4** In a clock auction, agent $i$ with value $v_i$ is said to ‘bid truthfully’ if he accepts clock price if and only if $p_i(h) > v_i$. (Equivalently, if the agent uses a cutoff strategy with cutoff $v_i^+ = \min \{p_i(h) : h \in H, p_i(h) > v\}$.)

**Definition 5** An auction is “weakly group strategy-proof” if for every profile of values $v$ and every set of players $S \subseteq N$ and every strategy profile $\sigma_S$ of these players, at least one bidder in $S$ has a weakly higher payoff from the profile of truthful bids $v_N$ than from the strategy profile $(v_{N\setminus S}^+, \sigma_S)$.

**Remark 1** Clock and threshold auctions are not generally “strongly” group strategy-proof, because a bid increase by a losing bidder that increases a winner’s threshold price is strictly profitable for the winner and weakly profitable for the loser.

Clock auctions can have various information disclosure policies, leading to a potentially large set of strategies for bidders, but always including the cutoff strategies. The definition of group strategy-proofness applies to all such auctions.

**Proposition 5** Every finite clock auction (with any information disclosure) is weakly group strategy-proof.

**Proof.** Every truthful bidder has a payoff of at least zero. Consider the first stage of clock auction affected by a group deviation. A bidder cannot benefit by a deviation that results in it losing and getting a deviation payoff of zero, so the first deviation must be made by a bidder who chooses not to exit at a
price equal to or below his truthful value, and who becomes winning. That bidder’s price, however, must then be below his truthful value, so that bidder loses from the group deviation.

The preceding argument is independent of the information policy in the auction, but it applies in particular when bidders are restricted to play cutoff strategies. Combining the two previous propositions, we get

**Corollary 1** Any DA threshold auction is weakly group strategy-proof.\(^{17}\)

### 7 Pay-as-Bid: Full-info equivalence

Recall that for any bid space \(B\) and allocation rule \(\alpha\), the rounded up values are \(v_i^+ = \min \{ b_i \in B_i : b_i > v \} \) and the threshold prices for winners are given by \(p_i (b_{-i}) = \max \{ b_i' \in B_i : i \in \alpha (b_i', b_{-i}) \} \). In particular, \(i \in \alpha (p_i (b_{-i}) , b_{-i}) \).

**Proposition 6** For every paid-as-bid DA auction with bid sets \(B_i\) and every value profile \(v\) with each \(v_i < \max B_i\), there is a complete-information Nash equilibrium bid profile \(b\) with \(b_i = \max \{ v_i^+, p_i (v_i^+) \} \) for each \(i \in N\). The corresponding equilibrium allocation is \(\alpha (b) = \alpha (v^+)\).

**Proof.** Since changing accepted bids so that they are still accepted does not affect the outcome of the DA algorithm, we have \(A \equiv \alpha (v^+) = \alpha \left( p_A (v^+) , v_{N\setminus A}^+ \right) = \alpha (b)\) and \(p_i (b_{-i}) = p_i \left( p_{A \setminus i} (v^+) , v_{N \setminus A}^+ \right) = p_i (v_{-i}^+) \geq v_i^+\) for each \(i \in A\). Now, we verify that the profile \(b\) is a Nash equilibrium. Every bidder \(i \in A\) is winning and receiving payment of \(b_i = p_i (v_i^+) = p_i (b_{-i}) \geq v_i^+\), and any larger bid by \(i\) would be losing, so a winning bidder \(i\) has no profitable deviation. Every bidder \(i \in N \setminus A\) is losing with its bid of \(v_i^+\), and so any winning bid for \(i\) earns a negative payoff: a losing bidder has no profitable deviation. \(\blacksquare\)

Next, we introduce a pair of standard definitions.

---

\(^{17}\)One can also establish a version of this result directly without restricting attention to finite bid spaces.
**Definition 6** An auction is dominance-solvable in state \( v \) if under full information, there exists a unique payoff profile that remains after iterated deletion of (weakly) dominated strategies, regardless of the order of elimination.

For “generic” values \( v_i \notin B_i \) for each \( i \), a unique payoff profile implies a unique outcome (allocation and winning bids).

For intuition, notice that iterated deletion of weakly dominated strategies closely resembles a deferred-acceptance procedure, because it works by iterated rejections using a myopic criterion and finally accepting all strategies that are not rejected. We will find below that, for paid-as-bid auctions, iterated elimination of weakly dominated bids is closely related to the iterated deletions of always losing bids that characterize DA auctions, and that this implies a similarly close connection between dominance solvable auctions and DA auctions.

**Definition 7** Assignment rule \( \alpha \) is non-bossy if for any \( i \in N \), \( b \in B \) and \( b_i' \in B_i \), \( \alpha (b_i', b_{-i}) \cap \{i\} = \alpha (b) \cap \{i\} \) implies \( \alpha (b_i', b_{-i}) = \alpha (b) \).

Non-bossiness means simply that a bidder cannot affect others’ allocations without changing his own allocation. Some DA algorithms are non-bossy, but as our “reference pricing” example illustrates, some are not. In a DA algorithm, a winner who changes its bid without changing its winning status (that is, agents \( i \in \alpha (b_i, b_i) \cap \alpha (b_i', b_i) \)) can never affect others’ winning status, but because bidder’s scores can depend on losing bids, a change by a loser to a different losing bid \( (i \notin \alpha (b_i, b_i) \cup \alpha (b_i', b_i)) \) could affect the set of winners.

**Proposition 7** Consider a paid-as-bid auction with a monotonic, non-bossy assignment rule \( \alpha \) and bid spaces \( B \). Say that a value profile \( v \) is “generic” if for each \( i \), \( v_i \in [0, \bar{v}_i] \setminus B_i \).

(i) The auction is pure-strategy dominance-solvable for all generic value profiles if and only if \( \alpha \) can be implemented with a DA algorithm.
(ii) If \( \alpha \) can be implemented with a DA algorithm, then for every generic value profile, the unique payoff profile surviving iterated deletion of dominated strategies is also the unique payoff profile associated with any (pure or mixed) Nash equilibrium in undominated strategies.

(iii) If \( \alpha \) can be implemented with a DA algorithm, then one strategy profile that survives iterated deletion of dominated strategies and is a Nash equilibrium in undominated strategies is the one described in Proposition 6.

**Remark 2** The equivalence of dominance solvability and implementation using a DA algorithm – equivalently, using the results of Section 5, a clock-auction algorithm – is constructive and intuitive. Indeed, a clock auction decrements the price to a bidder when the bidder could no longer win at that price, given the prices already accepted by the other bidders. Similarly, in a round of iterated deletion of dominated strategies in a paid-as-bid auction, a bid may be dominated because it always loses against all remaining possible bid profiles of the other bidders (when a lower bid, still above the bidder’s value, might sometimes be winning). The extra work in the proof arises because, in a paid-as-bid auction, a bid can also be dominated in two other ways: either it may be less than the bidder’s value or another, higher, bid may win against the same opposing bid profiles.

**Remark 3** We need \( \alpha \) to be non-bossy to guarantee a unique outcome because of examples like the following one (which could arise as a particular instance of yardstick competition described in Example 6). Let \( N = 2 \), \( B_1 = \{1, 3\} \), \( B_2 = \{2, 4\} \), \( \alpha(b) = \{1\} \) if \( b_2 = 4 \), and \( \alpha(b) = \emptyset \) otherwise. With \( v_1 < 3 \), bidder 1’s dominant strategy is to bid 3, but bidder 2 has no dominated strategies. The two strategy profiles that survive iterated elimination in this example are also the undominated Nash equilibrium profiles: they are (3, 4) (in which bidder 1 wins) and (3, 2) (in which there is no winner). In this example, iterated dominance and undominated Nash equilibrium fail to nail down the behavior of losing bidders in a paid-as-bid auction game: that is
what leads to a multiplicity of possible outcomes. When \( \alpha \) is non-bossy, there can still be many losing bids, but one can deduce the unique outcome just by reasoning about winning bids, as is done in the proof below.

**Proof.** For the “if” direction of (i), recall from Proposition 3 that any assignment rule \( \alpha \) that is implementable via a DA threshold auction is also implementable with a clock auction in which bidders use cutoff strategies with cutoffs corresponding to their sealed bids. Furthermore, we can implement assignment rule \( \alpha \) with paid-as-bid pricing using a “two-phase clock auction” mechanism, as follows. In phase 1, the clock auction algorithm described above is run to determine the set of winners. In phase 2, the payments to the winners are determined by allowing prices to continue falling (through points in \( B_i \)) until all bidders “quit”: each winning bidder is paid the last price it has accepted. In this two-phase clock auction game, if bidders use the cutoff profile \( b \), that obviously leads to the same outcome as the paid-as-bid DA auction game with bid profile \( b \).

If the assignment rule is non-bossy, then for generic values \( v_i \in \mathbb{R} \setminus B_i \) the game satisfies the TDI condition of Marx and Swinkels (1997), and so the payoff profiles surviving iterated deletion do not depend on the order of deletion: hence deleting dominated or equivalent strategies in any order leads to the same set of possible outcomes.\(^{18}\) We specify the following deletion process: Begin by deleting for each agent \( i \) all the bids/cutoffs \( b_i < v_i^+ \) (which are either dominated by or equivalent to the bid \( v_i^+ \)). In the game that remains after these initial deletions, every bidder strictly prefers any outcome in which it wins to any in which it loses. We specify the next deletions inductively by referring to the sequence of prices \( \{p(A^t)\} \) that would emerge

\(^{18}\)Given strategy sets \( \hat{B}_i \subseteq B_i \) for each \( i \), we say that two strategies \( b_i, b'_i \in \hat{B}_i \) are equivalent if \( \not\in \alpha (b_i, b_{-i}) = \alpha (b'_i, b_{-i}) \) for all \( b_{-i} \in \hat{B}_{-i} \). Obviously deleting strategies that are equivalent to surviving ones does not affect the solution to iterated deletion of dominated strategies. Note furthermore that given non-bossiness, such equivalence obtains whenever agent \( i \)'s own allocation does not change, that is, \( \alpha (b_i, b_{-i}) \cap \{i\} = \alpha (b'_i, b_{-i}) \cap \{i\} \) for all \( b_{-i} \in \hat{B}_{-i} \).
during phase 1 if each bidder were to use the cutoff strategy $v_i^+$. At the
beginning of each step $t$ of our iterated deletion process, the set of strategies
remaining to each bidder $i$ is $\hat{B}_i^{t-1} = B_i \cap [v_i^+, \max \{v_i^+, p_i(A^{t-1})\}]$. As the
prices are reduced to $p(A^t)$, for each bidder $i$, all the cutoffs $b_i \in \hat{B}_i^{t-1}$ such
that $b_i > \max \{v_i^+, p_i(A^t)\}$ are sure to lose and are therefore either domi-
nated by or equivalent to the cutoff $v_i^+$. Hence, we can eliminate strategies to
obtain $\hat{B}_i^t = B_i \cap [v_i^+, \max \{v_i^+, p_i(A^t)\}]$. These iterated deletions continue
until phase 1 ends at some iteration $T$. The winners are then determined to
be $\alpha(\max \hat{B}^T)$. For each agent $i$, if $\hat{B}_i^T$ is not a singleton, then its largest
element, $\max \hat{B}_i^T = \max(v_i^+, p_i(A^T))$, is dominant in the remaining game
with just the bids $\hat{B}^T$ (because it wins at the highest remaining price). So,
we may do one more round of deletions, taking $\hat{B}_i^{T+1} = \{\max(v_i^+, p_i(A^T))\}$. Hence, the only possible outcome of iterative elimination of undominated or
equivalent cutoffs in any order is the outcome corresponding to the bid profile
$(\max(v_i^+, p_i(A^T)))_{i \in N}$.

For (ii), fix an undominated mixed Nash equilibrium profile. For each
bidder $i$ with a zero equilibrium payoff, all bids of $v_i^+$ or more must be always
losing. Hence, by non-bossiness, we may replace every such bidder $i$’s bids
by the pure strategy bid $v_i^+$ to obtain another mixed strategy profile $\sigma$ with
the same distribution of outcomes. We show below that $\sigma$ is actually a pure
strategy bid profile, and specifically it is the profile $(\max(v_i^+, p_i(A^T)))_{i \in N}$
that results from the iterated eliminations described above.

For any bidder $i$ with strictly positive equilibrium expected payoffs, all
bids in the support of $\sigma_i$ have positive expected payoffs, so all must win
with a positive probability against $\sigma_{-i}$. Consider the maximum bid profile
in the support of $\sigma$. Referring to the clock auction process, we infer that if
any positive-payoff bidder’s bid is losing for that profile, then it is losing for
all profiles in the support of $\sigma$, which contradicts positive expected payoffs.
Since reducing a winning cutoff/bid in the clock auction does not affect the
allocation, for every bid profile in the support of $\sigma$, the positive-payoff players
are the winners. Since the highest always-winning bid earns strictly more than any lower winning bid, this further implies that the winners’ equilibrium mixtures are degenerate: winning bidders play pure strategies. Therefore, \( \sigma \) assigns probability one to some single bid profile \( b \).

Next, we claim that the iterative deletions described in the proof of (i) above do not delete any of the component bids in \( b \). Phase I of the iterative deletion procedure deletes only bids above \( v_i^+ \) for zero-payoff bidders and only always-losing bids for positive-payoff bidders, so all the component bids in \( b \) survive that phase. Phase II deletes all but the highest remaining bid of each winning bidder: the lower bids are never best replies to the highest surviving bids (they always win, but they are paid less). Hence, the full procedure never deletes any component bid in the profile \( b \). It follows that \( b = (\max(v_i^+, p_i(A_t)))_{i \in N} \) and that the outcome of \( b \) is the outcome of every undominated Nash equilibrium.

To prove (iii): in the surviving bid profile \( b \), each agent \( i \in A_T \) bids its threshold price, which is \( p_i(A_T) \leq v_i^+ \), while each \( i \in N \setminus A_T \) bids \( v_i^+ \), which is by definition above its threshold price. Thus by Proposition 6 it is a Nash equilibrium and it contains only undominated strategies, and as argued above it survives iterated deletion of dominated strategies.

It remains to prove the “only if” direction of (i): we assume that the auction is dominance solvable and relate a sequence of sets \( \hat{B}(A_t) \) surviving a number of rounds of iterated elimination of dominated or equivalent strategies to a corresponding sequence of clock prices \( p(A_t) \) that implements \( \alpha \). Importantly, our construction has the properties for each \( i \) and any possible history \( A_t \) of the auction that (a) \( \max \hat{B}_i(A_t) = p_i(A_t^{-1}) \) for every \( i \in A_t \), \( \min \hat{B}_i(A_t) > p_i(A_t^{-1}) \) for every \( i \in N \setminus A_t \), and \( \alpha(b) \subseteq A_t \) for all \( b \in \hat{B}(A_t) \), and (b) the strategy sets \( \hat{B}(A_t) \) are determined by iterated deletion of dominated and equivalent strategies in a particular order for any generic value profile \( v \) such that \( [v_i^+ \leq p_i(A_t^{-1})] \) if and only if \( i \in A_t \). We establish properties (a) and (b) by induction.
We initialize the construction with clock prices \( p(N) = \max B \) and sets of profiles \( \hat{B}(N) = B \). For each clock round \( t = 1, 2, 3, \ldots \), given any legal history \( A^t \) and previously determined strategy profiles \( \hat{B}(A^t) \), we build \( p(A^t) \) and \( \hat{B}(A^{t+1}) \) as follows. Within each “clock” iteration \( t \), we nest a second iteration employing a dummy variable \( \tilde{B} \). Initialize \( \tilde{B} = \hat{B}(A^t) \). Check whether there is some \( i \in A_t \) and \( b_i, b'_i \in \tilde{B} \) such that \( b'_i > b_i \) and \( \alpha(b'_i, b_{-i}) = \alpha(b_i, b_{-i}) \) for all \( b_{-i} \in \tilde{B}_{-i} \). If there is, we delete \( b_i \) from \( \tilde{B} \). Notice that the bid \( b_i \) is dominated by or equivalent to \( b'_i \) for all value profiles \( v \) (it wins against the same profiles \( b_{-i} \) and earns a higher price when it wins), so this step deletes only equivalent or weakly dominated strategies. Repeat this step to further trim \( \tilde{B} \) until the checking step indicates that no such qualifying bids \( b'_i > b_i \) remain.

We claim that, after maximal trimming of \( \tilde{B} \), either all remaining strategy profiles lead to the same winners (i.e. \( \alpha(\tilde{B}) = \{A_t\} \)) or else there exists an agent \( i \in A_t \) for whom the bid \( p_i(A^{t-1}) = \max \tilde{B} \) always loses, that is, \( i \in N \setminus \alpha(p_i(A^{t-1}), b_{-i}) \) for all \( b_{-i} \in \tilde{B}_{-i} \). To establish this claim, we use the inductive property and the assumption of dominance solvability for the game with a value profile \( v \) satisfying \( v^+_i < \min \tilde{B} \) for \( i \in A_t \) (so that agents \( i \in A_t \) always strictly prefer to win) and \( v^+_i > p_i(A^{t-1}) \) for \( i \in N \setminus A_t \) (the remaining agents prefer to lose). First, by inductive property (b), iterated deletion of dominated and equivalent strategies for \( v \) yields the sets \( \hat{B}(A^t) \). Next, given such sets, if there are any two bids \( b_i, b'_i \in \hat{B}_i(A^{t-1}) \) such that \( b'_i > b_i \) and \( b_i \) is dominated by \( b'_i \), then (by monotonicity) both win against the same set of opposing bid profiles \( b_{-i} \) and hence (by non-bossiness) lead to the same allocations. Bids \( b_i \) that are dominated in this way are eliminated by the iterative “pruning” described above, at the end of which none such remain in \( \tilde{B} \). Hence, unless there is a unique set of winners (\( \alpha(\tilde{B}) = \{A_t\} \)), dominance solvability for value profile \( v \) implies that there is another dominance relation to be found: there exists at least one active bidder \( i \in A_t \) and bids \( b_i, b'_i \in \hat{B}_i \) with \( b'_i < b_i \) such that \( b_i \) is dominated by \( b'_i \). Given \( v \), such dominance is
possible only if \( b_i \) never wins, which by monotonicity implies our claim that 
\( p_i (A^{t-1}) = \max \bar{B}_i \geq b_i \) never wins.

For iteration \( t \) of the clock auction, we reduce the price to the identified bidder \( i \) by letting 
\( p_i (A^t) = \max (\bar{B}_i \setminus \{ p_i (A^{t-1}) \}) \) and 
\( p_j (A^t) = p_j (A^{t-1}) \) for every bidder \( j \in N \setminus \{ i \} \). The clock auction and the strategy sets for the next round are then updated as follows. If bidder \( i \) accepts the reduced clock price 
\( p_i (A^t) \) at iteration \( t \), we let 
\( A_{t+1} = A_t \) and 
\( \hat{B}_i ((A^t, A_t)) = \hat{B}_i \setminus \{ p_i (A^{t-1}) \} \).
If, instead, bidder \( i \) quits, we let 
\( A_{t+1} = A_t \setminus \{ i \} \) and 
\( \hat{B}_i ((A^t, A_t \setminus \{ i \})) = \{ p_i (A^{t-1}) \} \). For all bidders \( j \in N \setminus \{ i \} \), regardless of \( i \)'s decision to accept or reject, we let 
\( p_j (A^t) = p_j (A^{t-1}) \) and 
\( \hat{B}_j ((A^t, A_t)) = \hat{B}_j ((A^t, A_t \setminus \{ i \})) = \bar{B}_j \).
This guarantees that property (b) extends to both history \((A^t, A_t)\) and 
\((A^t, A_t \setminus \{ i \})\).

To see that with this construction, property (a) also extends from \( t \) to \( t+1 \), observe that it suffices to check the property for the bidder \( i \) whose price is changed. If \( v_i^+ \leq p_i (A^{t-1}) \), then the bidder remains active and 
\( \max \hat{B}_i (A^{t+1}) = p_i (A^t) \), as specified by the inductive property. Otherwise, 
\( v_i^+ > p_i (A^{t-1}) \), the bidder quits and 
\( \hat{B}_i (A^{t+1}) = \{ p_i (A^t) \} \), so 
\( \min \hat{B}_i (A^{t+1}) > p_i (A^t) \). By (a), the clock auction with cutoffs \( b \) leads to the outcome \( \alpha (b) \).

Now we describe two classes of non-bossy allocation rules. One class contains efficient allocation rules (recall from Example 2 that in some cases such rules can be computed by DA heuristics):

**Lemma 1** Consider an efficient allocation rule \( \alpha \) given by (2) and assume that there are no ties (i.e. \( \arg\min \) is always single-valued). Then the allocation rule is non-bossy.

**Proof.** Note that

for \( i \notin \alpha (b_i, b_i) \cup \alpha (b'_i, b_i) \), (2) implies 
\( \alpha (b_i, b_i) = \arg \min_{A \in \mathcal{F} : i \notin A} \sum_{j \in A} b_j = \alpha (b'_i, b_i) \),
for \( i \in \alpha (b_i, b_i) \cap \alpha (b'_i, b_i) \), (2) implies 
\( \alpha (b_i, b_i) = \arg \min_{A \in \mathcal{F} \cap \mathcal{A}} \sum_{j \in A \setminus \{ i \}} b_j = \alpha (b'_i, b_i) \).
Another class contains allocation rules computed by DA heuristics with fixed scoring and perfect feasibility checking:

**Lemma 2** Suppose that $\mathcal{F} \subseteq 2^N$ is a comprehensive family of feasible sets (see Example 1), and that scoring is given by $s_i^A(b_i, b_{N\setminus A}) = \begin{cases} \sigma_i(b_i) & \text{if } A \setminus \{i\} \in \mathcal{F}, \\ 0 & \text{otherwise}, \end{cases}$ where $\sigma_i : B_i \to \mathbb{R}_{++}$ are strictly increasing functions that have no ties (so feasibility is always maintained). The allocation rule computed by the resulting DA heuristic is non-bossy.

**Proof.** As observed above, every DA procedure satisfies non-bossiness for the accepted bids. To check that condition for the rejected bids, too, we show that if given bid profile $b$ agent $i$’s bid is rejected in some round $t$, then replacing his bid with some $b'_i > b_i$ does not affect the final outcome of the algorithm. Note that it suffices to check situations in which the replacement results in $b'_i$ rejected prior to round $t$: Indeed, otherwise $b'_i$ will be rejected in round $t$ and the replacement will not affect the behavior of the algorithm. Suppose first that the replacement results in $b'_i$ being rejected in round $t - 1$, and thus does not affect the behavior of the algorithm prior to round $t - 1$. Letting $A_{t-1}$ be the set of accepted bids in round $t - 1$, and letting agent $j$ be the agent whose bid is rejected in round $t - 1$, we must have $A_{t-1} \setminus \{i, j\} \in \mathcal{F}$, and

$$\max_{k \in A_{t-1} \setminus \{i, j\} : A_{t-1} \setminus \{j, k\} \in \mathcal{F}} \sigma_k(b_k) < \sigma_i(b_i) < \sigma_j(b_j).$$

Using this and the comprehensiveness of $\mathcal{F}$, we obtain

$$\max_{k \in A_{t-1} \setminus \{i, j\} : A_{t-1} \setminus \{i, k\} \notin \mathcal{F}} \sigma_k(b_k) \leq \max_{k \in A_{t-1} \setminus \{i, j\} : A_{t-1} \setminus \{j, k\} \in \mathcal{F}} \sigma_k(b_k) < \sigma_j(b_j),$$

which implies that bid $b_j$ must be rejected in round $t$. Then after round $t$ the algorithm will be unaffected by the replacement of $b_i$ with $b'_i$. Iterating this argument, we see that any increase in agent $i$’s bid, resulting in it being rejected in some round $\tau < t$, will not affect the allocation. ■
8 Conclusion

The analysis and results reported in this paper were developed in response to questions arising from an engagement to advise the FCC on the design of its “incentive auction.” The possibility that some deferred acceptance algorithm could achieve nearly optimal results is of obvious importance for that application: it inspired simulations to identify the DA algorithms that work best for the FCC’s particular interference graph. The ability to implement any deferred acceptance threshold auction with a clock auction has been important both because such auctions may be easier for bidders than comparable sealed-bid auctions, and because a clock auction remains weakly group strategy-proof even if bidders do not trust the auctioneer’s computations. Strategy-proofness is also important because it reduces the cost of participation, especially for small local broadcasters whose participation is needed for a successful incentive auction. Our outcome equivalence results suggest that strategy-proofness can be introduced without raising acquisition costs compared to a paid-as-bid rule. Since there can be many Nash equilibria, the credibility of the outcome equivalence result is enhanced by the related outcome uniqueness result. In the incentive auction, revenues for the forward auction portion must be sufficient to pay the costs the broadcasters incur in moving to new broadcast channels, as well as meeting certain other gross and net revenue goals, so the possibility of including a cost target in the scoring rule is necessary to make the whole design possible. Including yardstick competition allows the FCC to set maximum prices for broadcasters in regions with little competition based on bids in other, more competitive regions.

One important limitation of our analysis is its focus on single-minded bidders. In practice, bidders are often interested in selling or buying various goods or bundles and have different values for those packages. For example, in the FCC’s “reverse problem,” in addition to selling their UHF stations to go off-air, bidders may also consider switching to a lower, less-congested
VHF band. Also, some broadcasters own multiple stations and may wish to contemplate which subset of stations to sell.

Deferred-acceptance auctions and clock auctions have also been used and studied for multidimensional bidders. For some special settings, DA auctions have been proposed that are strategy-proof (e.g. Kelso and Crawford 1987, Anubel 2004, Bikhchandani et al. 2011), but other kinds of proposed auctions are not. For example, in the simultaneous multiple-item ascending auctions studied by Milgrom (2000) and Gul and Stachetti (2000), bidders who may be willing to purchase more than one item at a time have an incentive to engage in monopsonistic “demand reduction.” Such incentives may be minimized in “large-market” settings in which there competition minimizes the power of each bidder to affect the prices he faces, ensuring approximate strategy-proofness. Another approach is to design an exactly strategy-proof clock auction that does not guarantee efficiency but only some approximation thereof (for an example, see Bartal et al. (2003). Further examination of DA auctions for multidimensional bidders remains an important direction for future research.

Roth (2002) has observed that “Market design involves a responsibility for detail, a need to deal with all of a market’s complications, not just its principle features.” Over the past two decades, variants of the original DA algorithm have had remarkable success in accommodating the diverse details and complications of a wide set of matching market design problems. In this paper, we extend that success to new class of auction design problems, connect DA auctions to encompass clock auctions, and reaffirm the deferred-acceptance idea as the basis for many of the most successful mechanisms of market design.
References


9 Appendix: Substitutable DA rules

We say that a clock auction implements allocation rule \( \alpha \) if \( \{ p_i(h) : h \in H \} = B_i \) for each \( i \) and the auction generates \( \alpha(b) \) when bidders use cutoff strategies with cutoffs \( b \in B \). Recall that allocation rule \( \alpha \) is monotonic if \( i \in \alpha(b) \) and \( b_i' < b_i \) implies \( i \in \alpha(b_i', b_{-i}) \). Say that allocation rule \( \alpha \) has substitutes if \( i \in \alpha(b) \) and \( b_j' > b_j \) for some \( j \neq i \) implies \( i \in \alpha(b_j, b_j') \). A set \( S \subseteq 2^N \) of subsets of \( N \) has no disposal if for all \( A, A' \in S \), \( A \subseteq A' \) implies \( A = A' \).

**Proposition 8** With finite bid spaces, any monotonic allocation rule with substitutes whose range has no disposal can be implemented with a clock auction or a deferred acceptance heuristic.

**Proof.** Given \( \alpha \), it can be implemented with a clock auction described as follows: Set \( p(N) = \max B \) and then in each period \( t \), set

\[
\begin{align*}
p_i(A^t) &= p_i\left(A^{t-1}\right)^- \text{ if } i \in A_t \setminus \alpha\left(p_{A_t} \left(A^{t-1}\right), p_{N \setminus A_t} \left(A^{t-1}\right)\right) \\
p_i(A^t) &= p_i\left(A^{t-1}\right) \text{ otherwise.}
\end{align*}
\]

(That is, decrement prices to those bidders who wouldn’t win given the current best offers – the current prices for the active bidders, and the last prices accepted by the bidders who have exited.)

To see that this auction implements \( \alpha \), observe that if bidders use cutoff strategies with cutoffs \( b_i \in B_i \), then a bidder \( i \in \alpha(b) \) can never exit the auction: when \( i \in A_t \) and he is offered price \( p_i(A^t) = b_i \), we will have \( p_{A_t \setminus \{i\}} \left(A^{t-1}\right) \geq b_{A_t \setminus \{i\}} \) and \( p_{N \setminus A_t} \left(A^{t-1}\right)^+ = b_{N \setminus A_t} \), hence the substitute property and \( i \in \alpha(b) \) imply \( i \in \alpha\left(p_{A_t} \left(A^{t-1}\right), p_{N \setminus A_t} \left(A^{t-1}\right)^+\right) \) and so his price is not decremented. Thus, we have \( \alpha(b) \subseteq A_t \) throughout the auction. On the other hand, when the auction stops we have \( A_t \subseteq \alpha\left(p_{A_t} \left(A^{t-1}\right), p_{N \setminus A_t} \left(A^{t-1}\right)^+\right) \), and putting together with the previous inclusion and using the no-disposal of the range of \( \alpha \) implies \( \alpha(b) = A_t = \alpha\left(p_{A_t} \left(A^{t-1}\right), p_{N \setminus A_t} \left(A^{t-1}\right)^+\right) \), hence the auction implements \( \alpha \). \( \blacksquare \)
Not all DA heuristics generate allocation rules that have substitutes. For example, the DA heuristic with the scoring rule
\[ s_1^{(1,2)}(b_1) = \max\{b_1 - 1, 0\}, \]
\[ s_2^{(2)}(b_2, b_1) = 1, \] and \[ s_2^{(1,2)}(b_2) = s_1^{(1)}(b_2, b_1) = 0 \] generates the allocation rule
\[ \alpha(b_1, b_2) = \begin{cases} 
1,2 & \text{if } b_1 < 1, \\
\emptyset & \text{otherwise}, 
\end{cases} \]
which does not have substitutes.

The following example shows that the no-disposal assumption is indispensable. Consider the allocation rule \[ \alpha(b) = \arg \min_{i \in N} b_i, \] and suppose \[ B_1 = ... = B_N \] (so that ties exist, and in case of ties all of them are winners). Then there is no clock auction to implement \( \alpha \). For any such auction would start with equal prices, and it would then not be “safe” to reduce any price: if all bidders have set their cutoff to the reserve, then the reduction would eliminate any affected bidder. On the other hand, if some one bidder has bid least, then failing to reduce the price prevents the algorithm from ever identifying that bidder.

The no-disposal assumption is satisfied, e.g., in Example 1, if the feasible set \( F \) is comprehensive (meaning that \( A \in F \) implies \( A' \in F \) whenever \( A \subseteq A' \)) and heuristic has perfect feasibility checking. But not all heuristic rules satisfy it (e.g., in Example 5, the heuristic may accept a set \( A' \subseteq N \) when bids are low and a set \( A \subseteq A' \) when bids are high).

Example 7 below exhibits a monotonic allocation rule whose range has no disposal that cannot be implemented by a clock auction, so the assumption of substitutes is not disposable.

10 Appendix: Comparisons to Properties of Other Auctions

The properties that we have derived for DA auctions are not shared by other classes of auctions that have received close attention. Below are some examples to show that our findings do not apply to auctions in which winners are selected using either optimization or a greedy-acceptance algorithm.
10.1 Optimizing Auctions

Consider the allocation rule $\alpha$ given by (2), which hat minimizes the total social cost subject to a feasibility constraint, and assume that the minimizer is unique. This allocation rule is a monotonic allocation rule, and if $B_i = (0, +\infty)$ then the corresponding threshold prices are Vickrey prices - the agent is paid the positive externality his participation creates on the total welfare other agents:

\[
p_i(b_{-i}) = \min_{A \subseteq F : A \subseteq N \setminus \{i\}} \sum_{j \in A} b_j - \sum_{j \in \alpha(b) \setminus \{i\}} b_j
\]

(so that his surplus $p_i(b_{-i}) - b_i$ captures the entire social cost savings due to his participation).

In some settings, such as the matroid case described in Example 2, the optimizing allocation rule and the Vickrey prices can be implemented with a DA algorithm or a clock auction. In this case, weak group strategy-proofness holds and there is payoff equivalence between Vickrey and paid-as-bid auctions also holds.\(^{19}\) In other settings, the optimizing allocation rule cannot be implemented with a DA algorithm and does not have the associated nice properties:

**Example 7** $N = \{1, 2, 3\}$ and $F = \{\{1, 2\}, \{3\}\}$. Intuitively, the structure of $F$ makes bidders 1 and 2 complementary. In this case, $\alpha(b) = \{1, 2\}$ if $b_1 + b_2 < b_3$ and $\alpha(b) = \{3\}$ if $b_1 + b_2 > b_3$. In any DA auction, the first bid to be rejected can be based only on pairwise comparisons of bids, so it cannot be generally consistent with the preceding inequalities.

Observe that the Vickrey auction implementing this allocation rule does

\(^{19}\)Bernheim and Whinston (1986) showed the paid-as-bid equivalence for the case of optimization in which bidders are substitutes, using coalition-proofness to select among Nash equilibria. Our analysis finds the same conclusion under different assumptions and conditions. We use either iterated dominance or undominated Nash equilibrium to select a Nash equilibrium and we allow a wide range of allocation rules with substitution, but we limit attention to environments with single-minded bidders.
not satisfy either weak group strategy-proofness or paid-as-bid equivalence. For example, when \( b_1 + b_2 < b_3 \), the Vickrey prices are \( p_1 (b_2, b_3) = b_3 - b_2 \) and \( p_2 (b_2, b_3) = b_3 - b_1 \). Then the two winners have a strictly improving coalitional deviation in which they bid \( b_1 < v_1, b_2 < v_2 \) such that \( b_1 + b_2 < v_3 \): both still win but each is paid strictly more. Also, note that the sum of Vickrey prices is \( 2b_3 - b_1 - b_2 > b_3 \), but in the corresponding paid-as-bid auction, there cannot be a Nash equilibrium in which bidders 1 and 2 win and are paid a total of \( b_1 + b_2 > v_3 \), since then bidder 3 would deviate to undercut them. (In fact, in any Nash equilibrium in which bidder 3 uses an undominated strategy and bidders 1 and 2 win, they together pay \( b_1 + b_2 = v_3 \). These Nash equilibrium outcomes have been characterized by Bernheim and Whinston (1986).) So, the Vickrey mechanism appears “too expensive” in this case relative to optimization with paid-as-bid pricing. One solution that has been proposed to the problem of Vickrey auctions’ excessive costs (insufficient revenues) is “core-selecting auctions” (Day and Milgrom 2008), which sacrifice strategy-proofness even for single-minded bidders. Deferred-acceptance threshold auctions offer a possible alternative way to reduce costs (increase revenues), which preserves strategy-proofness.

### 10.2 Greedy-Acceptance Auctions

To compare to the greedy-acceptance auctions of Lehmann, O’Callaghan and Shoham (2002), consider again Example 7. For illustration, let a bidder’s score be its bid, so the algorithm iterates accepting the highest bid that is still feasible. If we break ties in favor of lower-numbered agents, we have \( \alpha (b) = \{ b_1, b_2 \} \) if \( \min \{ b_1, b_2 \} \geq b_3 \), and \( \alpha (b) = \{ b_3 \} \) otherwise. Suppose bid spaces are \( b_i = [0, \bar{b}] \). The threshold payments for the reverse auction are as follows: First, if \( \alpha (b) = \{ b_1, b_2 \} \), then for \( i = 1, 2, p_i = \bar{b} \) if \( b_{-i} \leq b_3 \) and \( p_i = b_3 \) otherwise. Second, if \( \alpha (b) = \{ b_3 \} \), then \( p_3 = \min \{ b_1, b_2 \} \).

Observe that this allocation rule cannot be implemented with a DA algorithm or a descending clock auction, since the allocation is completely
determined by the single “best” (lowest) bid while the first step of the DA algorithm is determined by the single worst bid according to some criterion. One might conjecture that greedy-acceptance auctions could instead be matched with an ascending clock auction, but that fails, too, because when bidder 3 exits first, the allocation is determined to be \( \alpha (b) = \{b_1, b_2\} \) but the prices to the winners are not yet determined.\(^{20}\)

Next, observe that the greedy-acceptance threshold auction fails weak group strategy-proofness. For example, if \( v_1, v_2 > v_3 > 0 \), bidders 1 and 2 could jointly deviate to bid \( b_1, b_2 < v_3 \), which will give each of them threshold prices of \( \bar{b} \).

Finally, the threshold and paid-as-bid auctions based on the greedy-acceptance algorithm do not have the outcome-equivalence properties described in Section 7. For suppose that \( v_1, v_2 < v_3 \). In the threshold auction, bidders 1 and 2 win and their threshold payments are both \( \bar{b} \), but a paid-as-bid auction with complete information cannot have a pure Nash equilibrium in which bidders 1 and 2 win and both get paid above \( v_3 \), since then bidder 3 would deviate to undercut them both and win. So a greedy-acceptance algorithm with threshold payments is more expensive in this case than any pure Nash equilibrium of its paid-as-bid counterpart.

11 Appendix: Proof of Proposition 2

Let

\[
F_{Zn} = \{ S \subseteq N : (\exists c : S \rightarrow \{1, \ldots, n\})(\forall s, s' \in S)(\{s, s'\} \in Z \Rightarrow c(s) \neq c(s'))\}, \\
F_{mn}^d = \{ S \subseteq N : (\forall j) |S \cap N_j| \leq n - d\}, \text{ and } F_{Zmn}^d = F_{Zn} \cap F_{mn}^d.
\]

\(^{20}\)A traditional purpose of a clock auction is to economize on information transmission or conceal some information, and accordingly we require that that clock-auction prices stop changing once the allocation has been determined. Without this condition, any allocation rule can be implemented with a clock auction, simply by running all the prices down to elicit complete information about cutoffs from all bidders and determining the allocation as a function of those.
For each $S \subseteq N$, define $S_j = S \cap (\cup_{j' \leq j} N_{j'})$, $S'_j = S \cap N_j$, and $Z^*(S'_j) = \cap_{s \in S'_j} \{s' \in S_{j-1} : \{s, s'\} \in Z\}$ — i.e., the set of stations in $S_{j-1}$ that interfere with all of the stations in $S'_j$. If $Z^*(S'_j) \subseteq N_{j-1}$, then to avoid interference it is necessary to assign a different channel to each station in $S'_j \cup Z^*(S'_j)$. A necessary condition for this is $|S'_j \cup Z^*(S'_j)| = |S'_j| + |Z^*(S'_j)| \leq n$, which is assumption (ii) of the proposition. Less obviously, Hall’s Marriage theorem\(^\text{21}\) implies that this condition is also sufficient, allowing us to prove the following lemma.

**Lemma 3** $F_{Zmn}^d = F_{mn}^d$.

**Proof.** It is obvious that $F_{Zmn}^d \subseteq F_{mn}^d$ (the first set imposes all the same within-area constraints plus additional ones). For the reverse inclusion, consider any $S \in F_{mn}^d$. We will establish that $S \in F_{Zmn}^d$ by showing the possibility of constructing the required channel assignment function $c : S \rightarrow \{1, \ldots, n\}$.

Begin the construction by assigning a different channel $c(s)$ to each station $s \in S_1$, which is possible because $|S_1| \leq n$. Then, $c$ is feasible for $S_1$. Inductively, suppose that the channel assignment $c$ has been constructed to be feasible for stations $S_{j-1}$. We show how to extend $c$ to a feasible channel assignment for $S_j = S_{j-1} \cup S'_j$.

For any $S''_j \subseteq S'_j$, define $J(S''_j) = \{1, \ldots, n\} - c(Z^*(S''_j))$ and notice that $|J(S''_j)| = n - |c(Z^*(S''_j))|$. According to Hall’s Marriage Theorem (informally, think of the stations in $S'_j$ as the women and the $n$ channels as the men), there exists a one-to-one map $c : S'_j \rightarrow \{1, \ldots, n\}$ with the property that $(\forall s \in S'_j)c(s) \in J(\{s\})$ if and only if $(\forall S''_j \subseteq S'_j), |J(S''_j)| \geq |S''_j|$. Substituting for $|J(S''_j)|$, this last inequality is equivalent to $|c(Z^*(S''_j))| + |S''_j| \leq n$.

\(^{21}\)Hall’s Marriage Theorem concerns bipartite graphs linking two sets — “men” and “women.” Given any set of women $S$, let $A(S)$ be the set of men who linked (“acceptable”) to at least one woman in $S$. Hall’s theorem asserts that there exists a one-to-one match in which every woman is matched to some acceptable man (some men may be unmatched) if and only if for every subset $S'$ of the women, $|S'| \leq |A(S')|$. In our application, Hall’s theorem is used to show that channels can be assigned to the stations in each area without violating the channel constraints implied by the assignments in the lower indexed areas.
For all $S_j'' \subseteq S_j'$, since $S \in F_{mn}^d$, we have $|S_j''| \leq n - d$. Then, since $S_j'' \cup S_{j-1} \in F_{mn}^d$, it follows from assumption (ii) of the Proposition that $|S_j''| + |Z^*(S_j'')| \leq n$. Combining that inequality with $|c(Z^*(S_j''))| \leq |Z^*(S_j'')|$, we obtain the condition required by Hall’s Marriage theorem, implying the existence of a one-to-one function $c : S_j' \to \{1, \ldots, n\}$ such that $(\forall s \in S_j') c(s) \in J(\{s\})$. This extends $c$ to a feasible channel assignment for the expanded domain $S_j = S_j' \cup S_{j-1}$.

Finally, to establish the proposition, consider the following DA algorithm. At any round $t$, if the set of stations already assigned is $T$, then any station that is essential gets a score of zero. Among inessential stations at round $t$, the score for any station $s$ with $m(s) = j$ is $n - |T \cap N_j| + b(s)/(1 + b(s))$. (Intuitively, this algorithm tries to keep the number of stations assigned in each area roughly equal at every round.) By the above Lemma, this algorithm will first assign the most valuable station in each area, then the second most valuable station in each area, and so on until at least the $n - d$ most valuable stations in each area are assigned.