A note on the distribution of the aggregate claim amount at ruin

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Abstract

We consider the distribution of the aggregate claim amount at ruin in the classical risk model. We use probabilistic arguments to derive joint and marginal distributions involving the aggregate claim amount at ruin, as well as obtaining moments. We also discuss the use of a Gerber-Shiu-type function.

Keywords: classical risk model, time of ruin, deficit at ruin, aggregate claim amount at ruin

1 Introduction

In recent years, there have been many studies on ruin related quantities in insurance risk models. Notably, the paper by Gerber and Shiu (1998) has led to much research on quantities such as the time of ruin, the deficit at ruin and the number of claims until ruin. Whilst the probability of ruin remains the key focus of ruin theory, it is nevertheless of interest to study ruin related quantities and relationships between them.

Studies on the classical risk model show that the density of the time of ruin, given that ruin occurs, is positively skewed when the initial surplus is such that the ultimate ruin probability is non-negligible, say at least 0.5% (e.g. Dickson and Waters (2002)). Similarly, studies suggest that in the same circumstances the probability function of the number of claims until ruin, given that ruin occurs, is positively skewed (e.g. Egídio dos Reis (2002)). These results suggest that the distribution of the aggregate claim amount at ruin is likely to be positively skewed, and examples later in this paper suggest that this is indeed the case.

Whilst there have been various studies on the expected discounted amount of aggregate claims until the time of ruin (e.g. Cai et al. (2009), Feng (2009), Cheung (2013)), there is little in the literature on the aggregate claim amount at ruin. As we show in the next section, the aggregate claim amount at ruin is closely connected to the time of ruin and the deficit at ruin. Whilst the distribution of the time of ruin basically specifies the distribution of the insurer's income until ruin, the distribution of the deficit at ruin by itself tells us nothing about the distribution of an insurer's outgo until ruin. The joint distribution of the time of ruin and the deficit at ruin allows us to study the distribution of the aggregate claim amount until ruin, and properties such as moments. The distribution of the aggregate claim amount at ruin gives the insurer an indication of the likelihood of different levels of outgo should ruin occur. Rabehasaina and Tsai (2013) consider the joint distribution of the time of ruin and the aggregate claim amount at ruin in a Sparre Andersen process perturbed by diffusion. They use a Laplace transform approach, but are unable to invert the Laplace transform they obtain. Transform inversion is not an easy method for obtaining the distribution of ruin related quantities, except in the case when the initial surplus is zero – see, for example, Dickson and Willmot (2005) and Landriault et al. (2011). We take a much more straightforward approach than Rabehasaina and Tsai (2013) and obtain tractable expressions for the joint densities we study. Our aim is to give examples of joint densities involving the aggregate claim amount at ruin. We start our study by showing that the joint density of the time of ruin, the number of claims until ruin and the aggregate claim amount at ruin can be expressed in terms of the joint density of the time of ruin, the number of claims until ruin and the deficit at ruin.

This paper is set out as follows. In Section 2, we give notation, then in Section 3 we consider the joint distribution of the time of ruin, the number of claims until ruin and the aggregate claim amount at ruin. We consider moments of the aggregate claim amount at ruin in Section 4, discuss Gerber-Shiu analysis in Section 5, and make some concluding remarks in Section 6.

2 Notation

In the classical risk model an insurer's surplus is modelled as

$$U(t) = u + ct - S(t)$$

where u is the initial surplus, c is the rate of premium income per unit time, and $S(t)$ denotes the aggregate claim amount until time t. The aggregate claim amount is modelled as $S(t) = \sum_{j=1}^{N(t)} X_j$, where $\{N(t)\}_{t\geq 0}$ is a Poisson process with parameter λ and $\{X_j\}_{j=1}^{\infty}$ is a sequence of independent and identically distributed random variables, where X_j represents the amount of the jth claim. We use P, p and μ_k to respectively denote the distribution function, density function and kth moment of X_1, and we assume that $c > \lambda\mu_1$. Let $G(x,t) = \Pr(S(t) \leq x)$, with $g(x,t) = \frac{d}{dx}G(x,t)$ for $x > 0$.

We define the time of ruin as T_u so that $T_u = \inf\{t \,|\, U(t) < 0\}$ with $T_u = \infty$ if $U(t) > 0$ for all $t > 0$. The ultimate ruin probability is then $\psi(u) = 1 - \chi(u) =$

$\Pr(T_u < \infty)$. Let $Y_u = |U(T_u)|$ denote the deficit at ruin, $N(T_u)$ denote the number of claims until ruin (including the claim that causes ruin), and $S(T_u)$ denote the aggregate claim amount at the time of ruin. Let $v_n(u,y,t)$ denote the joint density of the time of ruin (t), the deficit at ruin (y) and the number of claims until ruin (n), and let $w_n(u,x,t)$ denote the joint density of the time of ruin (t), the aggregate claim amount at ruin (x) and the number of claims until ruin (n), with

$$v(u,y,t) = \sum_{n=1}^{\infty} v_n(u,y,t) \quad \text{and} \quad w(u,x,t) = \sum_{n=1}^{\infty} w_n(u,x,t)$$

respectively denoting the joint density functions of (T_u, Y_u) and $(T_u, S(T_u))$. For convenience, we later use the term *joint density* even if one of the variables we are considering is discrete.

Throughout we use the notation $\tilde{b}$ to denote the Laplace transform of a function b.

3 Joint densities involving $S(T_u)$

We now consider defective joint densities involving $S(T_u)$. Suppose that ruin occurs on the nth claim, at time t, and the deficit at ruin is $x - u - ct (> 0)$. Then the aggregate claim amount at ruin is x. Hence

$$\Pr(T_u \le t,\ N(T_u) = n \text{ and } S(T_u) \le x) = \int_0^t \int_0^{x-cs-u} v_n(u,y,s)\, dy\, ds$$

where $x > u + ct$ and $t > 0$ and

$$\begin{aligned} w_n(u,x,t) &= \frac{\partial^2}{\partial t\, \partial x} \Pr(T_u \le t,\ N(T_u) = n \text{ and } S(T_u) \le x) \\ &= v_n(u, x - ct - u, t), \end{aligned}$$

with$w(u,x,t) = v(u, x - ct - u, t)$. Thus, knowledge of the joint density of (T_u, Y_u) allows us to evaluate the joint density of $(T_u, S(T_u))$. Some explicit results for the the joint density of (T_u, Y_u) are given in Dickson (2007, 2008); see also Landriault and Willmot (2009).

To obtain $w_n(u,x,t)$ we use formulae for $v_n(u,y,t)$ given in Dickson (2012). We get $w_1(u,x,t) = \lambda e^{-\lambda t} p(x)$ for $u \ge 0$. Then for $n = 1, 2, 3, \ldots$

$$w_{n+1}(0,x,t) = e^{-\lambda t} \frac{\lambda^{n+1} t^n}{n!} \int_0^{ct} \frac{z}{ct} p^{n^*}(ct - z) p(z + x - ct) dz$$

and

$$\begin{aligned} w_{n+1}(u,x,t) &= e^{-\lambda t} \frac{(\lambda t)^n}{n!} \int_0^{u+ct} p^{n*}(z) \lambda p(x - z) dz \\ &\quad - c \sum_{j=1}^{n} \int_0^t e^{-\lambda s} \frac{(\lambda s)^j}{j!} p^{j*}(u + cs)\, w_{n+1-j}(0, x - u - cs, t - s)\, ds. \end{aligned}$$

We can use these expressions to obtain joint densities involving two variables. Summation over n yields the joint density of $(T_u\,, S(T_u))$, whilst integration over t yields the joint density of $(N(T_u)\,, S(T_u))$. Specifically, the joint density of $(T_u\,, S(T_u))$ is given by

$$w(0,x,t) = \lambda e^{-\lambda t}\, p(x) + \lambda \int_0^{ct} \frac{z}{ct}\, g(ct-z,t)\, p(z+x-ct)\, dz$$

and for $u>0$,

$$\begin{aligned} w(u,x,t) &= \lambda e^{-\lambda t}\, p(x) + \lambda \int_0^{u+ct} g(z,t)\, p(x-z)\, dz \\ &\quad - c\int_0^t g(u+cs,s)\, w(0,x-u-cs,t-s)\, ds. \end{aligned}$$

Similarly, the joint density of $(N(T_u)\,, S(T_u))$ is given by

$$\hat{w}_n(u,x) = \int_0^{(x-u)/c} w_n(u,t,x)\, dt\,. \tag{3.1}$$

These expressions can be evaluated for some claim size distributions, as illustrated later.

The density of $S(T_u)$, which we denote by ω, is most easily found as

$$\omega(u,x) = \int_0^{(x-u)/c} w(u,x,t)\, dt\,. \tag{3.2}$$

Using a joint density to obtain the marginal density of $S(T_u)$ may see like a convoluted approach. However, if we define a function

$$\Omega(u,x) = \int_0^x \omega(u,z)\, dz = \Pr(T_u < \infty \text{ and } S(T_u) \le x)$$

then, by using the standard technique of conditioning on the time and the amount of the first claim, we obtain

$$c\,\frac{d}{du}\Omega(u,x) = \lambda\,\Omega(u,x) - \lambda(P(x)-P(u)) - \lambda \int_0^u p(u-z)\,\Omega(z,x+z-u)\, dz.$$

The nature of the integral in this expression does not allow us to use standard approaches to solve for $\Omega(u,x)$ for claim size distributions for which explicit solutions for $\psi(u)$ exist. A second standard approach is transform inversion, which we discuss in Section 5. As we explain there, it is also not a promising route to obtain the density of $S(T_u)$.

3.1 Exponential claims

We consider here the case when $p(x) = \alpha e^{-\alpha x}$, $x > 0$. In this case it is well known (e.g. Gerber (1979)) that $v(u,x,t) = w(u,t)\,\alpha e^{-\alpha x}$, and so $w(u,x,t) = w(u,t)\,\alpha e^{-\alpha(x-ct-u)}$. Using formula (11) of Dickson (2007) for $w(u,t)$ we obtain the joint density of $(T_u\,, S(T_u))$ as

$$w(u,x,t) = \lambda\alpha\, e^{-\lambda t-\alpha x}\left(I_0\left(\sqrt{4\alpha\lambda t(u+ct)}\right) - \frac{ct}{ct+u} I_2\left(\sqrt{4\alpha\lambda t(u+ct)}\right)\right)$$

for $x > u + ct$ and $t > 0$, where

$$I_v(t) = \sum_{n=0}^{\infty} \frac{(t/2)^{2n+v}}{n!\,(n+v)!}$$

is the modified Bessel function of order v.

Figure 3.1 shows the conditional joint density of $(T_u\,, S(T_u))$ given that ruin occurs, for four different values of u, with the label t representing time of ruin and the label x representing the aggregate claim amount at ruin.

We can obtain the density of $S(T_u)$ from expression (3.2). Writing the Bessel functions in summation form and using the binomial expansion of $(u + ct)^n$ we obtain

$$\begin{aligned}\omega(u,x) \quad = \quad & \lambda\alpha\, e^{-\alpha x}\left(\sum_{n=0}^{\infty}\frac{(\alpha\lambda)^n}{n!n!}\sum_{j=0}^{n}\binom{n}{j}u^j c^{n-j}\,\frac{\Gamma(2n+1-j)}{\lambda^{2n+1-j}}\,E(2n-j+1,\lambda,(x-u)/c)\right.\\ & \left. -\sum_{n=0}^{\infty}\frac{(\alpha\lambda)^{n+1}}{n!\,(n+2)!}\sum_{j=0}^{n}\binom{n}{j}u^j c^{n+1-j}\,\frac{\Gamma(2n+3-j)}{\lambda^{2n+3-j}}\,E(2n-j+3,\lambda,(x-u)/c)\right)\end{aligned}$$

for $x > u$, where

$$E(m,\lambda,x) = 1 - \sum_{j=0}^{m-1} e^{-\lambda x}\,\frac{(\lambda x)^j}{j!}$$

is the Erlang(m,λ) distribution function.

Figure 3.2 shows the conditional density of $S(T_u)$ given that ruin occurs, for four different values of u. These plots exhibit the sort of shape that we would expect given that in the case of exponential claim sizes the distribution of T_u is positively skewed.

We can also obtain the joint density of $(N(T_u)\,, S(T_u))$. Using formula (3.1) we obtain

$$\hat{w}_n(0,x) = e^{-\alpha x}\,\frac{\alpha^n\, c^{n-1}}{\lambda^{n-1}}\,\frac{(2n-2)!}{n!\,(n-1)!}\,E(2n-1,\,\lambda,\,x/c)$$

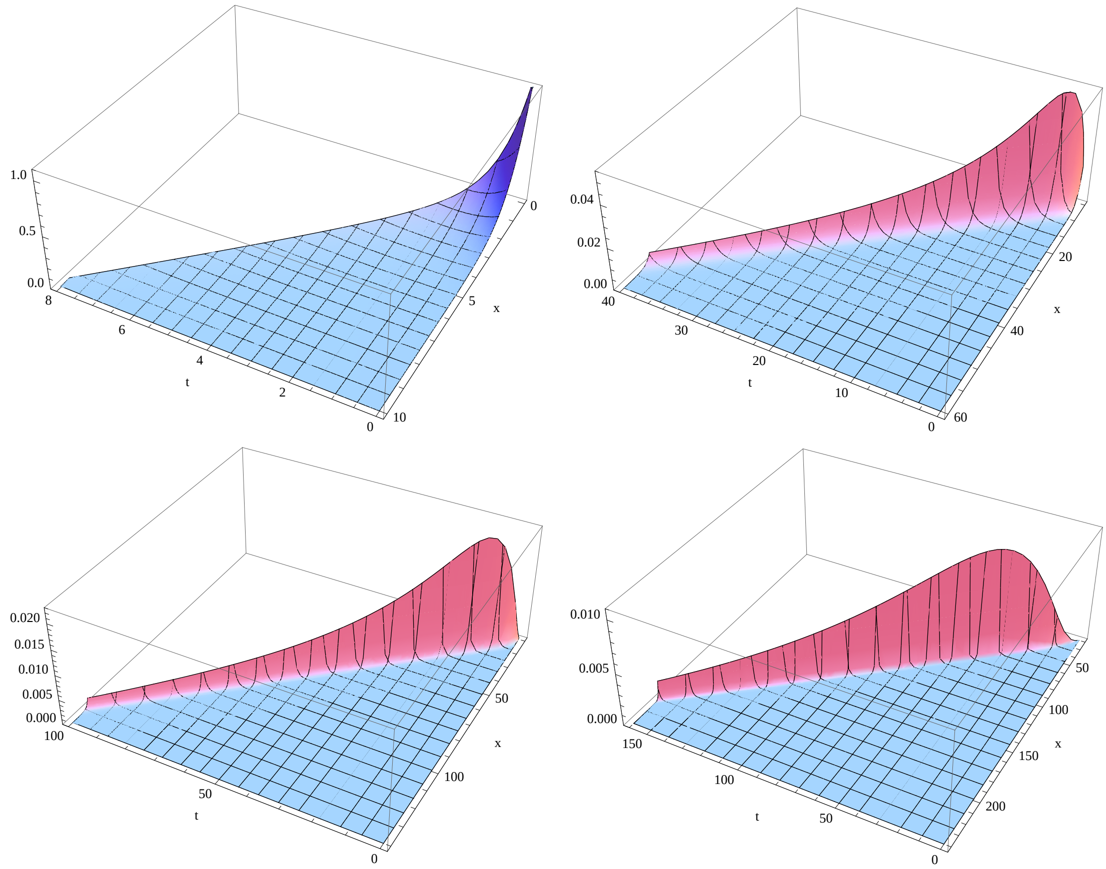

Figure 3.1: $w(u, x, t)$ for $u = 0, 5, 10, 20$ from top left to bottom right.

for $x > 0$. For $u > 0$, some straightforward manipulations lead to

$$\begin{aligned}
\hat{w}_n(u,x) \quad = \quad & \frac{\alpha^n\, u^{n-1}\, e^{-\alpha x}}{(n-1)!} \sum_{j=0}^{n-1} \left(\frac{c}{\lambda\, u}\right)^j \frac{(n+j-1)!}{j!\,(n-j-1)!}\, E(n+j-1,\, \lambda,\, (x-u)/c) \\
& - \left(\frac{\alpha\, c}{\lambda}\right)^n \frac{e^{-\alpha x}}{u} \sum_{k=1}^{n-1} \sum_{j=0}^{k-1} \left(\frac{\lambda\, u}{c}\right)^{k-j} \frac{(k+j)!\,(2n-2k-2)!}{k!\,(n-k)!\,(n-k-1)!\,j!\,(k-j-1)!} \\
& \times E(2n+j-k-1,\, \lambda,\, (x-u)/c)
\end{aligned}$$

for $x > u$.

Figure 3.3 shows plots of $\hat{w}_n(0, x)$ for $n = 2, 3, 4, 5$ (left hand plot) and $n = 10$ and $n = 20$ (right hand plot). We observe a positive skew for small values of n, but as n increases we see that the plots become much more symmetrical. For a given value of n, the area under the curve is $\Pr(N(T_0) = n)$, and so, for example, the area under the curve for $n = 10$ is much larger than in the case $n = 20$, since

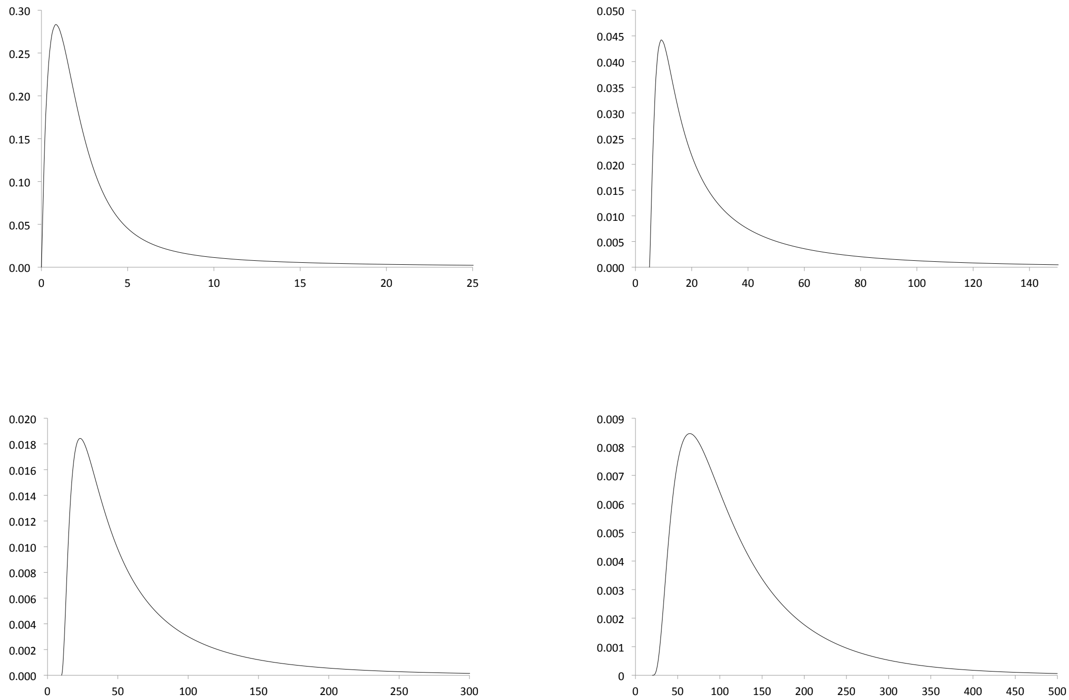

Figure 3.2: $\omega(u,x)$ for $u = 0, 5, 10, 20$ from top left to bottom right.

the probability function of $N(T_0)$ is a decreasing function in this case. Figure 3.4 shows plots of $\hat{w}_n(20,x)$ for $n = 2,3,4,5$ (left hand plot) and $n = 10$ and $n = 20$ (right hand plot). For the smaller values of n we observe the same sort of shapes as in Figure 3.3, although the pattern is reversed – the highest peak occurs for $n = 5$ in the left hand plot because in this case the probability function of the number of claims until ruin is initially increasing. We see that in the case $n = 20$ the plot is almost symmetric, and we observe this for higher values of n too. These plots suggest that the distribution of $S(T_u) \,|\, N(T_u)$ may be normal in this case. Although it is not apparent from our formula for $\hat{w}_n(u,x)$ why this would be so, we note that $S(T_u) \,|\, N(T_u)$ is a sum of random variables, so for reasonably large $N(T_u)$, we might expect a normal distribution.

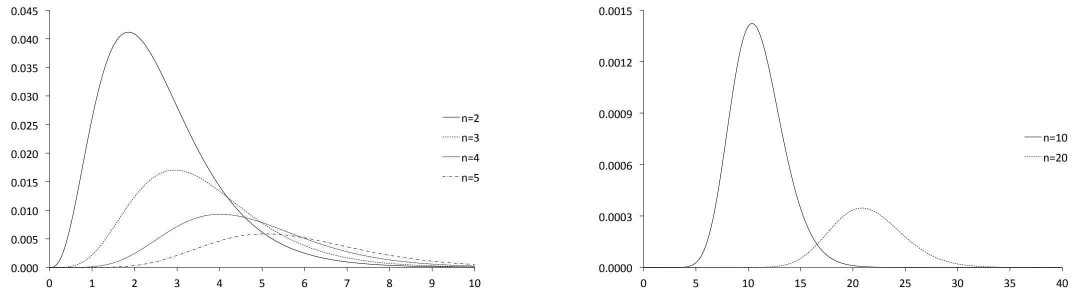

Figure 3.3: $\hat{w}_n(0, x)$ for some values of n

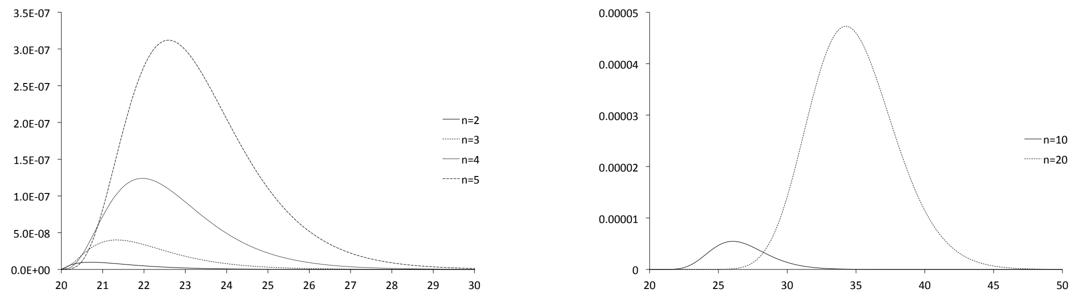

Figure 3.4: $\hat{w}_n(20, x)$ for some values of n

3.2 Other claim size distributions

Explicit solutions for the joint distribution of $(T_u\,, Y_u)$ exist for only a few individual claim size distributions. These distributions satisfy the property

$$p(x + y) = \sum_{j=1}^{m} \eta_j(x)\, \tau_j(y)$$

where $\{\eta_j\}$ are functions and $\{\tau_j\}$ are probability density functions. See Willmot (2007) for a discussion of this factorisation. We then have that $v(u, y, t)$ is of the form

$$v(u, y, t) = \sum_{j=1}^{m} h_j(u, t)\, \tau_j(y).$$

(See Dickson (2009).) Thus, we obtain

$$w(u,x,t) = \sum_{j=1}^{m} h_j(u,t)\,\tau_j(x-ct-u).$$

The general problem in implementing this formula is that it is difficult to obtain the functions $\{h_j(u,t)\}$, although this is not such a problem in the special case when $u=0$. For example, when $p(x) = \sum_{i=1}^{n} w_i\,\alpha_i e^{-\alpha_i x}$ we know from Dickson and Drekic (2007) that

$$\int_0^\infty e^{-\delta t}\,h_i(0,t)\,dt = \frac{\lambda}{c}\,\frac{w_i}{\alpha_i+\rho}$$

where ρ is the unique positive solution of Lundberg's fundamental equation

$$\lambda + \delta - cs = \lambda\,\tilde{p}(s)$$

(see Gerber and Shiu (1998)). Using the transform inversion relationship of Dickson and Willmot (2005) we then obtain

$$h_i(0,t) = \frac{\lambda w_i}{c}\left(c\,e^{-(\lambda+\alpha_i c)t} + \int_0^{ct}\frac{x}{t}\,g(ct-x,t)\,e^{-\alpha_i x}\,dx\right)$$

for $i = 1,2,\ldots,n$.

4 Moments of $S(T_u)$

By constructing a Gerber-Shiu-type function (as discussed in the next section), it is possible to obtain the moments of $S(T_u)$ using the approaches of either Lin and Willmot (2000) or Albrecher and Boxma (2005) to find the moments of T_u. However, we choose to find moments of $S(T_u)$ through the relationship

$$(u + cT_u - S(T_u))\;I(T_u<\infty) = -Y_u\,I(T_u<\infty) \tag{4.1}$$

where I is the indicator function. In particular, this yields

$$\begin{aligned} E[S(T_u)\,|\,T_u<\infty] &= c\,E[T_u\,|\,T_u<\infty] + E[Y_u\,|\,T_u<\infty] + u,\\ E[S(T_u)^2\,|\,T_u<\infty] &= c^2\,E[T_u^2\,|\,T_u<\infty] + 2c\,E[T_u\,Y_u\,|\,T_u<\infty]\\ &\quad + E[Y_u^2\,|\,T_u<\infty] + 2u\,E[S(T_u)\,|\,T_u<\infty)] - u^2. \end{aligned}$$

and

$$V[S(T_u)\,|\,T_u<\infty] = c^2\,V[T_u\,|\,T_u<\infty] + V[Y_u\,|\,T_u<\infty] + 2c\,\mathrm{Cov}[T_u,Y_u\,|\,T_u<\infty]\,.$$

This result is more evident by writing

$$(S(T_u)-u)\,I(T_u<\infty) = (cT_u+Y_u)\,I(T_u<\infty),$$

and we can use this to find higher moments of $S(T_u)$ recursively. To do this we require to evaluate expressions of the form $E[T_u^m\, Y_u^n\, I(T_u < \infty)]$ where m and n are positive integers, and we can do this using results given in Lin and Willmot (2000).

As noted earlier, when the individual claim amount distribution is exponential, the deficit at ruin is independent of the time of ruin. Thus, when $p(x) = \alpha e^{-\alpha x}$, $x > 0$, we have, using results from Lin and Willmot (2000),

$$E[S(T_u) \,|\, T_u < \infty] = \frac{2c\alpha - \lambda + c\alpha^2 u}{\alpha(c\alpha - \lambda)},$$

so that $E[S(T_u) \,|\, T_u < \infty]$ is linear in u, and

$$V[S(T_u) \,|\, T_u < \infty] = \frac{2c^3\alpha^3 + 2c^2\alpha^2\lambda(u\alpha - 1) + 3c\alpha\lambda^2 - \lambda^3}{\alpha^2(c\alpha - \lambda)^3},$$

which is also linear in u, but calculations with other claim size distributions show that this is not a general property.

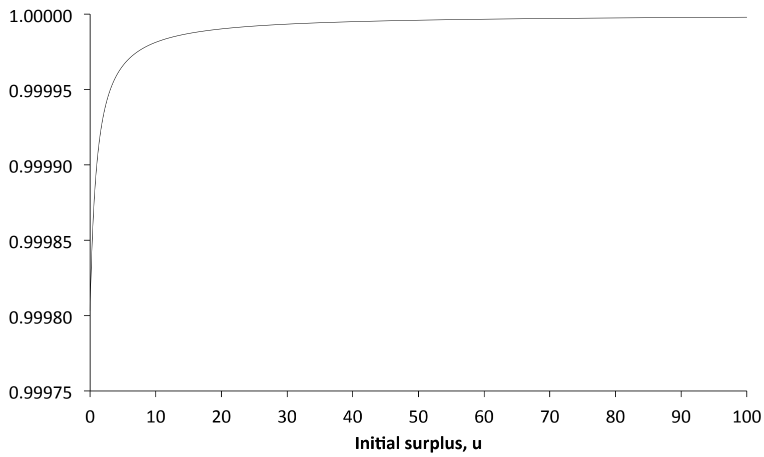

Figure 4.1: $\mathrm{Cov}[T_u,\, S(T_u) \,|\, T_u < \infty]$, exponential claims

The relationship (4.1) allows us to find the covariance between T_u and $S(T_u)$. If we multiply (4.1) by $T_u\, I(T_u < \infty)$, rearrange, and then take expectation, we obtain

$$\begin{aligned} E[T_u\, S(T_u)\, I(T_u < \infty)] &= c\, E[T_u^2\, I(T_u < \infty)] + E[Y_u\, T_u\, I(T_u < \infty)] \\ &\quad + u\, E[T_u\, I(T_u < \infty)]. \end{aligned} \tag{4.2}$$

As all the terms on the right hand side of (4.2) are required to find $V[S(T_u)]$, we know everything required to find $\mathrm{Cov}[T_u,\, S(T_u)\,|\, T_u < \infty]$. For example, when claims are exponentially distributed with mean 1, and when $\lambda = 1$ and $\theta = 0.1$, we have $\mathrm{Cov}[T_u,\, S(T_u)\,|\, T_u < \infty] = 110(21 + 20u)$ and the correlation coefficient between T_u and $S(T_u)$ given that $T_u < \infty$ is

$$\frac{231 + 220u}{\sqrt{53,382 + 101,660u + 48,400u^2}}\,.$$

Figure 4.1 shows a plot of this correlation coefficient as a function of u, and, unsurprisingly, this correlation coefficient is very close to 1 for all values of u. Further, as $\sqrt{48,400} = 220$, we see that it goes to 1 as $u \to \infty$.

5 Gerber-Shiu-type analysis

In this section we briefly discuss the application of a Gerber-Shiu-type function to problems involving the aggregate claim amount at ruin. We define a function $\phi_{\delta,r,a}(u)$ for $u \geq 0$ as

$$\phi_{\delta,r,a}(u) = E[e^{-\delta T_u - rS(T_u)} a^{N(T_u)} I(T_u < \infty)], \tag{5.1}$$

where $\delta \geq 0$, $r \geq 0$ and $0 < a \leq 1$. By setting $r = 0$ we obtain a function studied by Landriault et al. (2011). Using the standard technique of conditioning on the time and the amount of the first claim we obtain

$$\frac{d}{du}\phi_{\delta,r,a}(u) = \frac{\lambda+\delta}{c}\phi_{\delta,r,a}(u) - \frac{\lambda}{c}\int_0^u a e^{-rx}\phi_{\delta,r,a}(u-x)p(x)dx - \frac{\lambda}{c}\int_u^\infty a e^{-rx}p(x)dx\,, \tag{5.2}$$

from which we obtain the Laplace transform as

$$\tilde{\phi}_{\delta,r,a}(s) = \frac{c\phi_{\delta,r,a}(0) - \frac{\lambda a}{s}(\tilde{p}(r) - \tilde{p}(r+s))}{cs - (\delta+\lambda) + \lambda a \tilde{p}(r+s)}. \tag{5.3}$$

The denominator of this expression gives Lundberg's fundamental equation as

$$cs - (\lambda + \delta) + \lambda a \tilde{p}(r+s) = 0 \tag{5.4}$$

and it is a straightforward exercise to show that this equation has a unique positive solution, denoted ρ^*. As ρ^* is a zero of the denominator of (5.3) it is also a zero of the numerator, leading to

$$\begin{aligned} \phi_{\delta,r,a}(0) &= \frac{\lambda a}{c}\left(\frac{\tilde{p}(r) - \tilde{p}(r+\rho^*)}{\rho^*}\right) \\ &= \sum_{n=1}^{\infty} a^r \int_0^\infty \int_{ct}^\infty e^{-\delta t - rx}\, w_n(0,x,t)\, dx\, dt. \end{aligned} \tag{5.5}$$

By extending the transform inversion techniques in Dickson and Willmot (2005) and Landriault et al. (2011) we can invert (5.5) to obtain the formulae for $w_n(0,x,t)$ given in Section 3.1. The key to inversion is to define a function

$$p_r(x) = \frac{e^{-rx}\,p(x)}{\tilde{p}(r)}$$

for $r > 0$. We then define p_r^1 to be the equilibrium distribution of p_r, and this has Laplace transform

$$\tilde{p}_r^1(s) = \frac{\tilde{p}(r+s) - \tilde{p}(r)}{s\,\tilde{p}'(r)}\,,$$

so that

$$\phi_{\delta,r,a}(0) = \frac{-\lambda\,a}{c}\,\tilde{p}'(r)\int_0^\infty e^{-\rho^* x}\,p_r^1(x)\,dx\,.$$

If we set $a = 1$ and $\delta = 0$ in (5.1) then we obtain the Laplace transform of $S(T_u)$. However, inversion of this transform to obtain the density of $S(T_u)$ does not seem to be a straightforward task, even for simple forms of the claim size density. The simplest case is when $p(x) = \alpha e^{-\alpha x}$, and in this case we can solve equation (5.2) to obtain

$$E[e^{-rS(T_u)}\,I(T_u < \infty)] = \left(1 - \frac{R}{\alpha + r}\right)e^{-Ru}$$

where

$$R = \frac{1}{2c}\left(-\lambda - c\alpha - cr - \sqrt{((\alpha+r)c - \lambda)^2 + 4\lambda cr}\right)\,.$$

Inversion of this expression does not appear to be a straightforward task, even when $u = 0$.

As noted in Section 4, we can use the function $\phi_{\delta,r,a}(u)$ do obtain moments involving $S(T_u)$. The approach in Section 4 is generally more efficient than using the function $\phi_{\delta,r,a}(u)$, but if we want to evaluate expressions of the form $E[N(T_u)^m\,S(T_u)^n\,I(T_u < \infty)]$ for integer m and n, then the the approach in Section 4 does not apply and we need to use $\phi_{\delta,r,a}(u)$. In particular, suppose we want to find $E[N(T_u)\,S(T_u)\,I(T_u < \infty)]$. Then we have

$$E[N(T_u)\,S(T_u)\,I(T_u < \infty)] = -\frac{d}{da}\frac{d}{dr}\phi_{\delta,r,a}(u)\big|_{\delta=0,r=0,a=1}.$$

To obtain this we start by differentiating equation (5.3) with respect to both a and r. This yields

$$\begin{aligned}
&(cs - (\delta+\lambda) + \lambda a\tilde{p}(r+s))\frac{d}{da}\frac{d}{dr}\tilde{\phi}_{\delta,r,a}(s) \\
= \;& c\frac{d}{da}\frac{d}{dr}\phi_{\delta,r,a}(0) - \frac{\lambda}{s}\left(\tilde{p}'(r) - \tilde{p}'(r+s)\right) - \tilde{\phi}_{\delta,r,a}(s)\lambda\tilde{p}'(r+s) \\
& -\frac{d}{da}\tilde{\phi}_{\delta,r,a}(s)\lambda a\tilde{p}'(r+s) - \frac{d}{dr}\tilde{\phi}_{\delta,r,a}(s)\lambda\tilde{p}(r+s).
\end{aligned} \tag{5.6}$$

Setting $\delta = 0$, $r = 0$, $a = 1$ and $s = 0$ we obtain the solution for the case $u = 0$ as

$$
\begin{aligned}
-c\frac{d}{da}\frac{d}{dr}\phi_{\delta,r,a}(0)\big|_{\delta=0,r=0,a=1} &= E[N(T_0)\,S(T_0)\,I(T_0 < \infty)] \\
= \lambda\mu_1 \int_0^\infty E[N(T_u)I(T_u < \infty)]\,du &+ \lambda \int_0^\infty E[S(T_u)I(T_u < \infty)]\,du \\
+\lambda\mu_1 \int_0^\infty \psi(u)\,du + \lambda\mu_2. & \qquad (5.7)
\end{aligned}
$$

For the case $u > 0$, we use equation (5.6), condition on $\delta = 0$, $r = 0$ and $a = 1$, and then substitute using the well known result

$$
cs - \lambda + \lambda\tilde{p}(s) = \frac{c\chi(0)}{\tilde{\chi}(s)}
$$

into equation (5.6). We get

$$
\begin{aligned}
&\frac{d}{da}\frac{d}{dr}\tilde{\phi}_{\delta,r,a}(s)|_{\delta=0,r=0,a=1} = \frac{\tilde{\chi}(s)}{c\chi(0)}\Big[c\frac{d}{da}\frac{d}{dr}\tilde{\phi}_{\delta,r,a}(0)|_{\delta=0,r=0,a=1} + \lambda\mu_1\left(\frac{1-\tilde{q}(s)}{s}\right) \\
&+\lambda\mu_1\tilde{q}(s)\int_0^\infty e^{-su}E[N(T_u)\,I(T_u < \infty)]\,du + \lambda\tilde{p}(s)\int_0^\infty e^{-su}E[S(T_u)\,I(T_u < \infty)]\,du \\
&+\lambda\mu_1\tilde{q}(s)\int_0^\infty e^{-su}\psi(u)\,du\Big],
\end{aligned}
$$

where we define $q(x) = x\,p(x)/\mu_1$. We also let $Q(x) = \int_0^x q(y)\,dy$. Inversion yields

$$
\begin{aligned}
E[N(T_u)S(T_u)I(T_u < \infty)] &= \frac{\chi(u)}{c\,\chi(0)}E[N(T_0)\,S(T_0)\,I(T_0 < \infty)] \\
&- \frac{\lambda\mu_1}{c\chi(0)}\left(\int_0^u \chi(y)\,dy - \int_0^u \chi(u-x)\,Q(x)\,dx\right) \\
&- \frac{\lambda\mu_1}{c\chi(0)}\int_0^u E[N(T_x)I(T_x < \infty)]\,g(u-x)\,dx \\
&- \frac{\lambda}{c\chi(0)}\int_0^u E[S(T_x)I(T_x < \infty)]\,h(u-x)\,dx \\
&- \frac{\lambda\mu_1}{c\chi(0)}\int_0^u \psi(x)\,g(u-x)\,dx, \qquad (5.8)
\end{aligned}
$$

where $g(x) = \int_0^x q(x-y)\,\chi(y)\,dy$ and $h(x) = \int_0^x p(x-y)\,\chi(y)\,dy$. An expression for $E[N(T_x)I(T_x < \infty)]$ can be found in Dickson (2012) and we have obtained an expression for $E[S(T_x)I(T_x < \infty)]$ in Section 4.

In the case when claims are exponentially distributed with mean 1, $\lambda = 1$ and $c = 1.1$, we find that the $\mathrm{Cov}[N(T_u), S(T_u)\,|\,T_u < \infty] = 100(22 + 21u)$ and that the correlation coefficient between $N(T_u)$ and $S(T_u)$ given that $T_u < \infty$ is

$$
\frac{11\sqrt{5}\,(22 + 21u)}{\sqrt{(231 + 221u)(1271 + 1210u)}}\,.
$$

A plot of this is very similar to Figure 4.1 with the correlation coefficient going from 0.9987 when $u = 0$ to 0.9989 as $u \to \infty$.

6 Concluding Remarks

Equation (4.1) applies to other models. One example is the Sparre Andersen risk model. Another is the Markov-modulated risk model under which the rate of premium income per unit time is constant. As with the classical risk model, the general problem in using equation (4.1) to find joint distributions involving the aggregate claim amount at ruin is being able to find the joint distribution of the time of ruin and the deficit at ruin. One example of a Sparre Andersen model for which this is known is the Erlang risk model with exponential claims (see Dickson et al. (2005) or Landriault et al. (2011)), and we can easily replicate the analysis in Section 3.1. The joint distribution of the time of ruin and the deficit at ruin in the Markov-modulated risk model is a problem we will consider in a subsequent paper.

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