Durable Goods Monopoly with Stochastic Costs

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Abstract

I study the problem of a durable goods monopolist who lacks commitment power and whose marginal cost varies stochastically over time. When costs are time-invariant, the Coase conjecture holds: all buyers trade immediately at a low price and the market outcome is efficient. In contrast, with time-varying costs the monopolist serves the different types of consumers at different times and charges them different prices. When the distribution of consumer valuations is discrete, the monopolist exercises market power and there is inefficient delay. With a continuum of types, the monopolist cannot extract rents and the market outcome is efficient.

Keywords: durable goods, Coase conjecture, stochastic costs, dynamic games.

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1 Introduction

Consider a monopolist who produces a durable good and who cannot commit to a path of prices. For settings in which production costs do not change over time, Coase (1972) argued that this producer would not be able to sell at the static monopoly price. After selling the initial quantity, the monopolist has the temptation to reduce prices to reach consumers with lower valuations. This temptation leads the monopolist to continue cutting prices after each sale. Forward-looking consumers expect prices to fall, so they are unwilling to pay a high price. Coase conjectured that these forces would lead the monopolist to post an opening price arbitrarily close to marginal cost. The monopolist would then serve the entire market “in the twinkling of an eye”, and the market outcome would be efficient. The classic papers on durable goods monopoly (i.e., Stokey, 1981, and Gul, Sonnenschein and Wilson, 1986) provide formal proofs of the Coase conjecture: as the period length goes to zero, the monopolist’s opening price converges to the lowest consumer valuation. In the limit, all buyers trade immediately, the market outcome is efficient and the monopolist is unable to extract additional rents from those buyers with higher valuations.

The purpose of this paper is to study the problem of a durable goods monopolist who lacks commitment power and whose cost of production varies stochastically over time. The assumption that costs are subject to stochastic shocks is natural in many markets. Time-varying costs may arise as a result of changes in input prices. For instance, high-tech firms face uncertain and time-varying costs, partly because the prices of some of their key inputs tend to fall over time, and partly due to fluctuations in the prices of the raw materials that they use. Changes in exchange rates will also lead to time-varying costs if the monopolist sells an imported good or if she uses imported inputs. The results in this paper show how such variations in costs can affect the dynamics of prices, the timing of sales and the seller’s profits in durable goods markets.

The model is set up in continuous time and the monopolist’s marginal cost evolves as a diffusion process. Costs are publicly observable, and at each moment the monopolist can produce any quantity at the current marginal cost. Continuous time methods are especially suitable to perform the option value calculations that arise with time-varying costs, allowing me to obtain a tractable characterization of the equilibrium. The model delivers simple expressions for the prices at which buyers are willing to trade, allowing for the computation of profit margins as a function of costs and the level of market penetration.

With time-varying costs, serving the entire market immediately is in general not efficient.

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1See Conlon (2010) for evidence on the evolution of production costs in the LCD TV industry.
The reason for this is that time-varying costs introduce an option value of delaying trade. The efficient outcome in this setting is that the monopolist serves consumers with valuation $v$ the first time costs fall below a threshold $z_v$. The threshold $z_v$ is decreasing in the valuation $v$, so under the efficient outcome the monopolist serves buyers sequentially as costs decrease. Based on this observation, I propose the following generalization of the Coase conjecture for markets with time-varying costs. Given a distribution of consumer valuations, I define an outcome to be Coasian if (i) it is Pareto efficient, and (ii) the monopolist is unable to extract additional rents from consumers with higher valuations. Note that the second condition does not require the monopolist to sell to all consumers at the same time or at the same price; in fact, doing so would violate efficiency.

In this paper, I show that the market outcome fails to be Coasian when costs are time-varying and the distribution of consumer valuations is discrete. With discrete types, the monopolist is able to extract rents from consumers with higher valuations, since she can truthfully commit to delay trade with buyers with lower types until costs are low enough. Moreover, there is inefficient delay in equilibrium. In contrast, the market outcome is Coasian when there is a continuum of types. In this case, the monopolist has an incentive to serve the next buyer arbitrarily soon after her last sale. As a result, the monopolist is no longer able to extract rents from consumers with higher valuations, and the market outcome is efficient.

Coase’s original arguments illustrate how commitment problems may prevent a monopolist producer of a durable good from exercising market power. The results in this paper show that these forces are more general than what Coase described. In particular, these forces do not rely on serving the entire market immediately, nor on serving every buyer at the same price. In markets with time-varying costs, to attain efficiency and zero rent extraction it is enough that the monopolist cannot credibly commit to delay trade from one sale to the next.

To see how time-varying costs modify the results on the Coase conjecture, consider a setting with two types of buyers: high types with valuation $v_H$, and low types with valuation $v_L < v_H$. After high types buy and leave the market, the monopolist’s problem is to choose when to sell to the remaining low type buyers. When costs do not change over time, it is optimal for the monopolist to sell to low types immediately after selling to high types. This is the force behind the Coase conjecture: high types are not willing to pay a high price, since they expect prices to fall rapidly after they buy. With time-varying costs, the monopolist will only sell to low types when costs fall below a threshold $z_L$. High types know that it will take a non-negligible amount of time for prices to fall when costs are above $z_L$, so they are willing to pay a higher price. In a sense, time-varying costs endogenously provide commitment power to the monopolist.
The equilibrium dynamics with two types of buyers are as follows. If costs are initially above a threshold $\bar{x} > z_L$, the monopolist first sells to all high type buyers, and then sells to low types when costs fall below $z_L$. If costs are below a threshold $\underline{x} < z_L$, the monopolist sells immediately to high and low types and the market closes. When costs lie between $\underline{x}$ and $\bar{x}$ the monopolist sells to high types gradually over time and market penetration increases continuously. The equilibrium is inefficient when costs lie in this intermediate range, since under the first-best outcome all high types trade immediately when costs are in this region.

Finally, I study markets with a continuum of valuations by analyzing a sequence of discrete models that approximates a model with a continuum of types. I show that the equilibrium outcome becomes Coasian when the type space becomes continuous. That is, despite the fact that different types of consumers pay different prices and buy at different times, with a continuum of types the outcome is efficient and the seller is unable to extract rents from consumers with higher valuations. Intuitively, the monopolist looses all commitment power when the gap between valuations becomes vanishingly small, since she now has an incentive to serve the next buyer arbitrarily soon after her last sale. As a result, she is no longer able to extract rents and the equilibrium outcome is efficient.

1.1 Related literature

The literature on durable goods monopoly has identified different ways in which a dynamic monopolist can exercise market power. For instance, a durable goods monopolist can ameliorate her lack of commitment power by renting her good rather than selling it (Bulow, 1982), or by introducing best-price provisions (Butz, 1990). The Coase conjecture also fails when the monopolist faces capacity constraints (Kahn, 1986, and McAfee and Wiseman, 2008), or when consumers use non-stationary strategies (Ausubel and Deneckere, 1989). The current paper studies the problem of a durable goods monopolist whose production costs change over time and identifies a new setting in which such a seller can exercise market power: time-varying costs provide commitment power to the monopolist when types are discrete, allowing her to extract rents from buyers with higher valuations.\(^2\)

This paper relates to Biehl (2001), who studies a durable goods monopoly model in other papers study dynamic monopoly models in non-stationary environments. Stokey (1979) solves the full commitment path of prices of a durable good monopolist when costs fall deterministically over time. Sobel (1991) studies the problem of a dynamic monopolist in a setting in which new consumers enter the market each period. Board (2008) characterizes the full commitment strategy of a durable good monopolist when incoming demand varies over time. Garrett (2012) solves the full commitment strategy of a durable good monopolist in a setting in which buyers arrive over time and their valuations are time-varying.

\(^2\)Other papers study dynamic monopoly models in non-stationary environments. Stokey (1979) solves the full commitment path of prices of a durable good monopolist when costs fall deterministically over time. Sobel (1991) studies the problem of a dynamic monopolist in a setting in which new consumers enter the market each period. Board (2008) characterizes the full commitment strategy of a durable good monopolist when incoming demand varies over time. Garrett (2012) solves the full commitment strategy of a durable good monopolist in a setting in which buyers arrive over time and their valuations are time-varying.
which the valuations of the buyers are subject to stochastic shocks that are independent across the population (see also Deb, 2011). Biehl (2001) shows that the monopolist may charge a constant high price in this setting. Intuitively, changes in valuations lead to a renewal of high type consumers, allowing the seller to truthfully commit to a constant high price. In my model, time-varying costs lead to changes in the net valuations of the buyers (i.e., valuations minus marginal cost). However, with changes in costs the net valuations of all buyers move simultaneously and in the same direction, leading to an entirely different equilibrium dynamics.

This paper also shares some features with models of bargaining with one-sided incomplete information (Fudenberg, Levine and Tirole, 1985). Deneckere and Liang (2006) study a bargaining game in which the valuation of the buyer is correlated with the cost of the seller (see also Evans, 1989, and Vincent, 1989). They show that there are recurring bursts of trade in equilibrium, with short periods of high probability of agreement followed by long periods of delay. In the current paper, there are also recurring bursts of trade when types are discrete. For instance, with two types of buyers the monopolist first sells to all high types when costs are initially large, and then sells to low types when costs fall below $z_L$.

Fuchs and Skrzypacz (2010) study a one-sided incomplete information bargaining game in which a new trader may arrive according to a Poisson process. The payoffs that the seller and the buyer get upon an arrival depend on the buyer’s valuation for the seller’s good; for instance, upon arrival the seller may run a second price auction between the original buyer and the new trader. Fuchs and Skrzypacz (2010) show that the seller is unable to extract rents in this setting: her inability to commit to a path of offers drives her profits down to her outside option of waiting until the arrival of a new buyer. Moreover, the possibility of arrivals leads to inefficient delays, with the seller slowly screening out high type buyers. In the current paper, the monopolist is also unable to extract rents when there is a continuum of types. However, the equilibrium outcome is efficient in this setting, with the seller serving the different buyers at the point in time that maximizes total surplus.

The possibility of arrivals in Fuchs and Skrzypacz (2010) introduces interdependencies in the net valuations of the buyer and the seller, making their setting similar to the one in Deneckere and Liang (2006). In a different paper, Fuchs and Skrzypacz (2013) consider the model in Deneckere and Liang (2006) with a continuum of types. They show that the equilibrium of this model converges to the outcome in Fuchs and Skrzypacz (2010) as the gap between the seller’s lowest cost and the buyer’s lowest valuation converges to zero. The seller is therefore unable to extract rents from the buyer in this gapless limit. In the current paper, the monopolist also looses the ability to extract rents as valuations become a continuum.
The difference, however, is that this result holds for any model with a continuum of types, regardless of the lowest consumer valuation, i.e., regardless of the size of the gap.\(^3\)

## 2 Model

A monopolist faces a continuum of consumers indexed by \(i \in [0, 1]\). Consumers are in the market to buy one unit of the monopolist’s good. Time is continuous and consumers can make their purchase at any time \(t \in [0, \infty)\). The valuations of the consumers are defined by a non-increasing and left-continuous function \(f : [0, 1] \rightarrow [\underline{v}, \overline{v}]\) with \(\overline{v} > \underline{v} > 0\); consumer \(i\)’s valuation for the good is \(f(i)\). Moreover, \(f\) is right-continuous at 0. Since \(f\) is non-increasing, consumers with a lower index have a weakly higher valuation. Consumers and the monopolist are risk-neutral expected utility maximizers and discount future payoffs at rate \(r > 0\). I assume that \(f\) is a step function taking \(n\) values \(v_1, \ldots, v_n\), with \(v_1 < \ldots < v_n\). For \(k = 1, \ldots, n\), let \(\alpha_k = \max\{i \in [0, 1] : f(i) = v_k\}\); i.e., \(\alpha_k\) is the highest indexed consumer with valuation \(v_k\). Section 6 considers the case in which \(f\) approximates a continuous function.

Let \(B = \{B_t, \mathcal{F}_t : 0 \leq t < \infty\}\) be a standard Brownian motion on a probability space \((\Omega, \mathcal{F}, P)\).\(^4\) The monopolist’s marginal cost \(x_t\) evolves as a geometric Brownian motion,

\[
    dx_t = \mu x_t dt + \sigma x_t dB_t, \tag{1}
\]

with \(x_0 > 0, \sigma > 0\) and \(|\mu| < r\). Note that this implies that \(x_t > 0\) for all \(t \geq 0\). The constants \(\mu\) and \(\sigma\) measure the expected rate of change and the percentage volatility of \(x_t\), respectively. At any time \(t\) the monopolist can produce any quantity at marginal cost \(x_t\). The assumption that \(\mu < r\) guarantees that the monopolist will always produce on demand: under this condition it is never optimal for the monopolist to produce when costs are low to sell in the future when costs are high. On the other hand, the assumption that \(\mu > -r\) guarantees that the monopolist will serve buyers in finite time: if \(\mu < -r\), the seller would always find it optimal to wait for costs to fall further and would therefore never make a sale. The process \(x_t\) is publicly observable and its underlying structure is common knowledge: seller and buyers commonly know that \(x_t\) evolves as (1).

A (stationary) strategy for consumer \(i \in [0, 1]\) is a function \(p : \mathbb{R}_+ \rightarrow \mathbb{R}_+\) that describes

\(^3\)More broadly, this paper also relates to the growing literature that uses continuous time methods to analyze strategic interactions. For instance, continuous time methods have been used to study the provision of incentives in dynamic settings (Sannikov, 2007, 2008), political campaigns (Gul and Pesendorfer, 2012) and dynamic markets for lemons (Daley and Green, 2012).

\(^4\)The filtration \(\{\mathcal{F}_t : 0 \leq t < \infty\}\) is the completion of the filtration generated by the Brownian motion.
the price that \( i \) is willing to pay for the good given any level of costs. Suppose consumer \( i \) is still in the market at time \( t \). Then, under strategy \( p(\cdot) \) consumer \( i \) purchases the good at time \( t \) if and only if the price that the monopolist charges is weakly lower than \( p(x_t) \).

Let \( \mathbf{P} = P(x,i) \) be a strategy profile for the consumers, with \( P(\cdot,i) \) denoting the strategy of consumer \( i \in [0,1] \). In any equilibrium, the strategy profile \( P(x,i) \) must satisfy the **skimming property**: for all \( i < j \), \( P(x,i) \geq P(x,j) \) for all \( x \). That is, buyers with higher valuations (i.e., buyers with lower indices) are willing to pay higher prices. The reason for this is that it is more costly for buyers with higher valuation to delay their purchase: if a buyer with valuation \( v \) finds it weakly optimal to buy at some time \( t \) given a future path of prices, then buyers with valuation \( v' > v \) find it strictly optimal to buy at \( t \). For technical reasons, I restrict attention to strategy profiles such that \( P(x,i) \) is left-continuous in \( i \) and continuous in \( x \). This restriction guarantees that payoffs are well defined.

The skimming property implies that at any time \( t \) there exists a cutoff \( q_t \in [0,1] \) such that consumers \( i \leq q_t \) have already left the market, while consumers \( i > q_t \) are still in the market. The cutoff \( q_t \) describes the level of market penetration at time \( t \). At each time \( t \), the level of market penetration and the monopolist’s marginal cost describe the payoff relevant state of the game. The initial level of market penetration \( q_0^- \) is equal to 0.

Given a strategy profile \( \mathbf{P} \), the monopolist chooses a path of prices to maximize her profits. Since \( \mathbf{P} \) satisfies the skimming property, by setting a price \( p \) the monopolist effectively chooses the level of market penetration: if the monopolist sets price \( p \) at time \( t \), there will be a \( q \in [0,1] \) such that \( P(x_t,i) \geq p \) if and only if \( i \leq q \). Moreover, the monopolist will find it optimal to charge a price \( P(x_t,q) \) if consumer \( q \) is the marginal buyer at time \( t \). Thus, I can alternatively specify the monopolist’s problem as choosing a non-decreasing process \( \{q_t\} \) with \( q_0^- = 0 \) and \( q_t \leq 1 \) for all \( t \), describing the level of market penetration at any time \( t \). With this specification, under strategy \( \{q_t\} \) the seller charges \( P(x_t,q_t) \) at every time \( t \), and at this price all buyers \( i \leq q_t \) who are still in the market buy.

**Remark 1** Throughout the paper, I maintain the standard assumption that the monopolist cannot ration consumers: at every point in time the monopolist sells to all buyers who are willing to purchase at the current price. This assumption implies that a strategy \( \{q_t\} \) must be such that, for all \( t \), \( q_t \geq i \) for all \( i \) with \( P(x_t,i) \geq P(x_t,q_t) \). Under this restriction, if consumer \( i \) buys at time \( t \) and if \( P(x_t,j) = P(x_t,i) \), then consumer \( j \) also buys at time \( t \).

**Monopolist’s problem:** Given a strategy profile \( \mathbf{P} \) of the consumers, a strategy for the seller is an \( \mathcal{F}_t \)-progressively measurable process \( \{q_t\} \) satisfying the conditions in Remark 1 such that \( q_0^- = 0 \), \( q_t \) is non-decreasing with \( q_t \leq 1 \) for all \( t \), and \( \{q_t\} \) is right-continuous with
left-hand limits.\(^5\) Let \(\mathcal{A}^P\) denote the set of all such processes. Given a strategy profile \(P\) of the consumers and a strategy \(\{q_t\} \in \mathcal{A}^P\), the monopolist’s profits are

\[
\Pi = E \left[ \int_0^\infty e^{-rt} (P(x_t, q_t) - x_t) dq_t \right]. \tag{2}
\]

Let \(\Pi (x, q)\) denote the monopolist’s future discounted profits conditional on the current state being \((x, q)\), and let \(\mathcal{A}^P_{q,t}\) denote the set of processes in \(\mathcal{A}^P\) such that \(q_t^- = q\). Then, the monopolist’s payoffs conditional on the state at time \(t^-\) being \((x, q)\) are\(^6\)

\[
\Pi (x, q) = \sup_{\{q_t\} \in \mathcal{A}^P_{q,t}} E \left[ \int_t^\infty e^{-r(s-t)} (P(x_s, q_s) - x_s) dq_s \bigg| \mathcal{F}_t \right]. \tag{3}
\]

Condition (3) is the requirement that the monopolist’s strategy \(\{q_t\}\) is time-consistent, since the strategy \(\{q_t\}\) must be optimal at every point in time.

**Consumer’s problem:** Given a strategy \(\{q_t\}\) of the monopolist and a strategy profile \(P\) of the consumers, the path of prices is \(\{P(x_t, q_t)\}\). The strategy \(P(x, i)\) of each consumer \(i\) must be optimal given the path of prices \(\{P(x_t, q_t)\}\): the payoff that consumer \(i\) gets from buying at the time strategy \(P(x, i)\) tells her to buy must be weakly larger than what she would get from purchasing at any other point in time. Formally, for any time \(t\) before consumer \(i\) buys, it must be that

\[
f(i) - p \leq \sup \{q_t \in \mathcal{A}^P_{q,t} \} E\left[ e^{-r(\tau-t)} (f(i) - P(x_\tau, q_\tau)) \bigg| \mathcal{F}_t \right]
\]

for all \(p > P(x, i)\), and

\[
f(i) - p \geq \sup \{q_t \in \mathcal{A}^P_{q,t} \} E\left[ e^{-r(\tau-t)} (f(i) - P(x_\tau, q_\tau)) \bigg| \mathcal{F}_t \right]
\]

for all \(p \leq P(x, i)\).

I impose two additional conditions on the strategies of the consumers. First,

\[
\forall i \text{ such that } f(i) = v_1, P(x, i) = v_1 \text{ for all } x.
\]

In words, all consumers with the lowest valuation are willing to pay a price equal to their valuation. The second condition I impose is as follows. Fix a strategy profile \(P\) of the consumers and a strategy \(\{q_t\}\) of the monopolist. Recall that \(\alpha_k\) is the highest indexed consumer with valuation \(v_k\). For \(k = 1, \ldots, n\), let \(\tau(k)\) denote the (possibly random) time at which the monopolist starts selling to consumers with valuation \(v_k\), i.e., \(\tau_k = \inf\{t : q_t > v_k\}\),

\(^{5}\)These continuity requirements on \(\{q_t\}\) together with the continuity requirements on \(P(x, i)\) guarantee that the integrals in (2) and (3) are well-defined.

\(^{6}\)Note that the profits \(\Pi (x, q)\) are conditional on the state at time \(t^-\). The reason for this is to preserve the right-continuity of \(\{q_t\}\), since this process may jump at time \(t\).
\( \alpha_{k-1} \}. \) Then, for \( k = 2, \ldots, n, \)

\[
v_k - P(x, \alpha_k) = E \left[ e^{-r(\tau_k-1-t)} \left( v_k - P(x_{\tau_k-1}, q_{\tau_k-1}) \right) \right] \bigg| \mathcal{F}_t, \tag{5}
\]

whenever the state at \( t \) is \( (x, \alpha_k) \). Equation (5) is an indifference condition: it states that the price consumer \( \alpha_k \) is willing to pay must leave her indifferent between buying at that price or waiting and buying at the price at which the monopolist starts selling to consumers with valuation \( v_{k-1} \). The paragraph below provides a justification for conditions (4) and (5).

**Definition 1** A strategy profile \((P, \{q_t\})\) is an equilibrium if:

(i) \( \{q_t\} \) is optimal for all states \((x, q) \in \mathbb{R}^+ \times [0, 1]\) given \( P \),

(ii) for each \( i \in [0, 1], P(x, i) \) is optimal given \( \{q_t\} \) and \( P \), and

(iii) \( P \) satisfies conditions (4) and (5) given \( \{q_t\} \).

Conditions (i) and (ii) in Definition 1 require that the strategies of the seller and the buyers be optimal. Condition (iii), on the other hand, imposes additional restrictions on the buyers’ strategies. In the supplementary appendix I show that the restrictions that condition (iii) imposes on the buyers’ strategies would necessarily hold in any subgame perfect equilibrium (SPE) of a discrete time version of this model. Intuitively, the reason for this is as follows: in discrete time durable goods monopoly games buyers incur a fixed cost of delay if they choose not to buy at the current price, since they must wait one time period for the seller to post a new price. This fixed cost of delay imposes strong restrictions on the strategies that buyers use in a SPE; in particular, this delay cost implies that the buyers’ strategies must satisfy the restrictions in condition (iii). In contrast, buyers do not face a fixed cost of delay when the game is in continuous time, since they can accept a new price within an arbitrarily short period of time. As a result, without imposing condition (iii) in the definition of equilibrium, the continuous time game would have equilibria that could never be the limiting SPE (as the time period goes to zero) of its discrete time counterpart.7 8

For instance, the strategy profile under which the monopolist always sells at a price equal to marginal cost, and under which each consumer buys at the time that maximizes her payoff (given that the monopolist always sells at price \( x_t \)) satisfies conditions (i) and (ii) in Definition 1, but doesn’t satisfy condition (iii). Note that the outcome induced by this strategy profile could never be the limiting SPE outcome (as the time period goes to zero) of a discrete-time version of this model, since in such a discrete-time game the buyers’ strategies would satisfy the restrictions in condition (iii).

7 The fact that, without additional restrictions, the continuous time game has equilibria that can never arise in discrete time is closely related to the multiplicity of equilibrium outcomes that arises in continuous time bilateral bargaining games in which there are no restrictions on the timing of offers and counteroffers; see, for instance, Bergin and MacLeod (1993) and the discussion in Perry and Reny (1993), pp. 66-67.
3 First-best outcome

This section computes the first-best outcome. Recall that the function $f$ describing the valuations of the consumers is a step function taking values $v_1 < \ldots < v_n$. To compute the efficient outcome, consider first the problem of choosing the surplus maximizing time at which to serve a homogeneous group of buyers with valuation $v_k$. This problem is given by

$$V_k(x) = \sup_{\tau \in T} E\left[ e^{-r\tau} (v_k - x) \mid x_0 = x \right],$$

where $T$ is the set of stopping times. Let $\lambda$ be the negative root of $\frac{1}{2}\sigma^2\lambda (\lambda - 1) + \mu \lambda = r$, (i.e., $\lambda = \frac{\sigma^2 - 2\mu - \sqrt{(2\mu - \sigma^2)^2 + 8\sigma^2}}{2\sigma^2}$), and for $k = 1, \ldots, n$ let $z_k = \frac{\lambda}{1-\lambda} v_k$.

**Lemma 1** The stopping time $\tau_k = \inf \{ t : x_t \leq z_k \}$ solves (6). Moreover,

$$V_k(x) = \begin{cases} (v_k - z_k) \left( \frac{x}{z_k} \right)^\lambda & x > z_k, \\ v_k - x & x \leq z_k. \end{cases} \quad (7)$$

**Proof.** Supplementary appendix. ■

Lemma 1 captures the option value that arises when costs vary stochastically over time. The total surplus from serving a group of buyers with valuation $v_k$ is maximized by waiting until costs fall below the threshold $z_k$. The threshold $z_k$ is increasing in $\mu$ and decreasing in $\sigma$, so it is optimal to wait longer when costs fall faster or when they are more volatile. By Lemma 1, the first-best outcome is that the monopolist serves consumers with valuation $v_k$ at time $\tau_k$. For instance, if $x_0 > z_n$, under the first-best outcome the monopolist serves buyers with valuation $v_n$ the first time $x_t = z_n$. After that, the monopolists serves buyers with valuation $v_{n-1}$ when $x_t = z_{n-1}$, and so on. Proposition 1 summarizes these results.

**Proposition 1** Under the first-best outcome, the monopolist serves buyers with valuation $v_k$ at time $\tau_k = \inf \{ t : x_t \leq z_k \}$.

4 Markets with two types of consumers

In this section, I study markets with two types of buyers: high types with valuation $v_2$, and low types with valuation $v_1 \in (0, v_2)$. Let $\alpha \in (0, 1)$ be the fraction of high type buyers, so that $f(i) = v_2$ for all $i \in [0, \alpha]$ and $f(i) = v_1$ for all $i \in (\alpha, 1]$. Section 4.1 derives
the equilibrium for such markets, and Section 4.2 presents the most salient features of the equilibrium.

4.1 Equilibrium

By equation (4), consumers with valuation \( v_1 \) are willing to pay a price equal to \( v_1 \). That is, for all \( i \in (\alpha, 1] \) and for all \( x \), \( P(x, i) = v_1 \). For any \( q \geq \alpha \), let \( \Pi(x, q) \) denote the monopolist’s profits when the level of market penetration is \( q \) and costs are \( x \). Note that at such a state only consumers with valuation \( v_1 \) remain in the market. Since all consumers with valuation \( v_1 \) buy at the same time, at any state \( (x, q) \) with \( q \geq \alpha \), the problem of the monopolist is to optimally choose the time at which to sell to all consumers remaining in the market; that is, \( \Pi(x, q) = (1 - q) \sup \mathbb{E}[e^{-r\tau} (v_1 - x_{\tau}) | x_0 = x] \). By Lemma 1, the solution to this problem is \( \tau_1 = \inf \{ t : x_t \leq z_1 \} \), so \( \Pi(x, q) = (1 - q) V_1(x) \) for all \( q \in [\alpha, 1] \).

Consider next the case in which the level of market penetration is \( q \in [0, \alpha) \), so there are \( \alpha - q \) high types remaining in the market. To study equilibrium at these states, I proceed in two steps. First, I establish a lower bound on the monopolist’s profits. Then, I show that the monopolist’s equilibrium profits are exactly equal to this lower bound.

Consider the strategy \( P(x, \alpha) \) of consumer \( \alpha \), the highest indexed consumer with valuation \( v_2 \). After consumer \( \alpha \) buys and leaves the market, the monopolist will sell to low type buyers the first time costs fall below the threshold \( z_1 \), and will charge them a price equal to \( v_1 \). Therefore, by equation (5), in any equilibrium \( P(x, \alpha) \) must satisfy

\[
P(x, \alpha) = v_2 - \mathbb{E}[e^{-r\tau_1} (v_2 - v_1) | x_0 = x].
\]

That is, when costs are equal to \( x \), consumer \( \alpha \) must be indifferent between buying at price \( P(x, \alpha) \) or waiting until costs fall below \( z_1 \) and obtaining the good at price \( v_1 \). Equation (8) highlights the commitment power that time-varying costs provide to the monopolist. When \( x_t > z_1 \), consumer \( \alpha \) knows that the monopolist won’t lower her price to \( v_1 \) until costs fall below \( z_1 \), so she is willing to pay strictly more than \( v_1 \). See Figure 1 for a plot of \( P(x, \alpha) \).

**Lemma 2** \( P(x, \alpha) - x > V_1(x) \) for all \( x \in (z_1, z_2] \). Moreover,

\[
P(x, \alpha) = \begin{cases} v_2 - (v_2 - v_1) \left( \frac{x}{z_1} \right)^\lambda & x > z_1, \\ v_1 & x \leq z_1, \end{cases}
\]

where \( \lambda \) is the negative root of \( \frac{1}{2} \sigma^2 \lambda (\lambda - 1) + \mu \lambda = r \).
Figure 1: Strategy $P(x,\alpha)$ of consumer $\alpha$; $v_1 = {1\over 2}$, $v_2 = 1$, $\mu = -0.02$, $\sigma = 0.2$ and $r = 0.05$.

**Proof.** Supplementary appendix. \hfill \blacksquare

Since the strategy profile of the buyers satisfies the skimming property, it must be that $P(x,i) \geq P(x,\alpha)$ for all $x$ and all $i < \alpha$. This implies that at any time $t$ such that $q_t < \alpha$, the monopolist can sell to all remaining high types at price $P(x_t, \alpha)$. Therefore, for all states $(x,q)$ with $q \in [0,\alpha)$, the monopolist’s profits are bounded below by

$$L(x,q) = \sup_{\tau \in T} \mathbb{E} \left[ e^{-\tau r} \left[ (\alpha - q) \left( P(x_\tau, \alpha) - x_\tau \right) + \Pi(x_\tau, \alpha) \right] \right] | x_0 = x,$$

where $T$ is the set of stopping times. At states $(x,q)$ with $q < \alpha$ the seller can pursue the following strategy: she can choose optimally the time $\tau$ at which to sell to the remaining high type buyers at price $P(x_\tau, \alpha)$, obtaining $(\alpha - q)(P(x_\tau, \alpha) - x_\tau)$ from these sales plus a continuation payoff of $\Pi(x_\tau, \alpha)$. By following this strategy, the seller earns $L(x,q)$.\footnote{In the discrete time version of this game, the monopolist’s profits are also bounded below by $L(x,q)$ in the limit as the time period goes to zero. The proof of this result is available upon request.} The following result characterizes the solution to the optimal stopping problem (10).

**Lemma 3** For every $q \in [0,\alpha)$, there exists $\underline{x}(q) \in (0,z_1)$ and $\overline{x}(q) \in (z_1,z_2)$ such that $\tau(q) = \inf\{ t : x_t \in [0,\underline{x}(q)] \cup [\overline{x}(q),z_2] \}$ solves (10). Moreover, $\underline{x}(\cdot)$ and $\overline{x}(\cdot)$ are continuous, with $\lim_{q \to \alpha} \underline{x}(q) = \lim_{q \to \alpha} \overline{x}(q) = z_1$.

**Proof.** Supplementary appendix. \hfill \blacksquare

To gain intuition behind the solution to (10), let $g(x,q) = (\alpha - q)(P(x,\alpha) - x) + \Pi(x,\alpha)$, so that $L(x,q) = \sup_{\tau \in T} \mathbb{E} \left[ e^{-\tau r} g(x_\tau,q) \right] | x_0 = x$. Since $P(x,\alpha)$ has a convex kink at $z_1$ (see
Figure 2: Lower bound to profits $L(x,q)$; $v_1 = \frac{1}{2}$, $v_2 = 1$, $\alpha = 0.7$, $\mu = -0.02$, $\sigma = 0.25$ and $r = 0.05$.

Figure 1) and since $\Pi(x,\alpha) \in C^1$, $g(x,q)$ also has a convex kink at $z_1$. Therefore, when $x$ is
around $z_1$ (i.e., when $x \in (\underline{x}(q), \bar{x}(q))$) the monopolist can obtain larger profits by delaying
trade with high type buyers than by serving all of them at price $P(x,\alpha)$ (see Figure 2).

Intuitively, the monopolist cannot sell to all remaining high types at a price significantly
larger than $v_1$ when costs are in $(\underline{x}(q), \bar{x}(q))$, since high types expect prices to fall fast
to $v_1$ after they all buy and leave the market. However, the monopolist has the option of
waiting and selling to the remaining high type buyers at a future point in time. Since the
price $P(x,\alpha)$ is increasing in $x$, the monopolist would be able to charge a higher price to
the remaining high type consumers if costs went up while she waited. The cutoffs $\underline{x}(q)$ and
$\bar{x}(q)$ are such that the monopolist gets a larger expected payoff by waiting than by selling to
all remaining high types immediately whenever $x \in (\underline{x}(q), \bar{x}(q))$. The solution to (10) also
involves delaying when costs are above $z_2$: serving high types is too expensive when $x > z_2$, so in this case it is optimal to wait for costs to fall.

The proof of Lemma 3 shows that, for all $x \in (\underline{x}(q), \bar{x}(q))$, the function $L(x,q)$ solves
the ordinary differential equation

$$rL_x(x,q) = \mu x L(x,q) + \frac{1}{2} \sigma^2 L_{xx}(x,q).$$  (11)

The general solution to the ordinary differential equation (11) is $L(x,q) = Ax^\lambda + Bx^\kappa$, where
$\lambda < 0$ and $\kappa > 1$ are the roots of $\frac{1}{2} \sigma^2 \lambda (\lambda - 1) + \mu \lambda = r$, and $A$ and $B$ are constants.
There are four unknowns: the constants $A$ and $B$, and the thresholds $\underline{x}(q)$ and $\bar{x}(q)$. The
four equations that determine these unknowns are the following value matching and smooth pasting conditions:

\[
L(x(q), q) = g(x(q), q), \quad L(\overline{x}(q), q) = g(\overline{x}(q), q), \quad (VM)
\]

\[
L_x(x(q), q) = g_x(x(q), q), \quad L_x(\overline{x}(q), q) = g_x(\overline{x}(q), q). \quad (SP)
\]

The optimal stopping problem (10) is defined for all \( q \in [0, \alpha) \): for each \( q \in [0, \alpha) \) there are cutoffs \( x(q) \) and \( \overline{x}(q) \) such that the solution to (10) involves stopping the first time \( x_t \in [0, x(q)] \cup [\overline{x}(q), z_2] \). Lemma 3 shows that \( x(\cdot) \) and \( \overline{x}(\cdot) \) are continuous, with \( \lim_{q \to \alpha} x(q) = \lim_{q \to \alpha} \overline{x}(q) = z_1 \). In words, the delay region \((x(q), \overline{x}(q))\) shrinks as \( q \) increases, and it disappears as \( q \to \alpha \). The intuition behind this result is as follows. The seller benefits by delaying trade with high types when \( x \in (x(q), \overline{x}(q)) \): in this region, the profits she gets by waiting are higher than what she would get if she sold to all high types immediately. However, this delayed trade with high types comes at a cost to the seller, since it implies that she will also be delaying trade with low types beyond what is optimal. Note that the gains from delaying trade decrease as there are fewer high types in the market, while the costs remain constant. As a result of this, the delay region \((x(q), \overline{x}(q))\) shrinks as \( q \) increases.

The following Theorem shows that the monopolist’s equilibrium profits are exactly equal to the lower bound \( L(x, q) \).

**Theorem 1** There exists a unique equilibrium. In equilibrium, at every state \((x, q)\) with \( q \in [0, \alpha) \) the monopolist’s profits are \( L(x, q) \). Moreover, for all \( t \geq 0 \) with \( q_t < \alpha \),

(i) if \( x_t > z_2 \), the monopolist doesn’t sell, so \( dq_t = 0 \),

(ii) if \( x_t \in [\overline{x}(q_t - ), z_2] \), the monopolist sells to all remaining high type consumers at price \( P(x_t, \alpha) \), so \( dq_t = \alpha - q_t - \),

(iii) if \( x_t \leq x(q_t - ) \), the monopolist sells to all remaining consumers (high and low types) at price \( v_1 \), so \( dq_t = 1 - q_t - \),

(iv) if \( x_t \in (x(q_t - ), \overline{x}(q_t - )) \), the monopolist gradually sells to high type consumers at price \( P(x_t, q_t) = x_t - L_q(x_t, q_t) \), so \( q_t \) is continuous and strictly increasing.

**Proof.** Appendix A.1. ■

Theorem 1 shows that the monopolist’s profits are equal to the lower bound \( L(x, q) \) for every state \((x, q)\) with \( q \in [0, \alpha) \). When \( x_t \in [\overline{x}(q_t - ), z_2] \), the monopolist sells to all
remaining high types at price \( P(x_t, \alpha) \), and then sells to low types when costs fall below \( z_1 \). When \( x_t \leq \underline{x}(q_t-) \), the monopolist sells to both low and high type buyers at price \( v_1 \) and the market closes. When \( x_t > z_2 \), the monopolist waits for costs to decrease. Finally, when \( x_t \in (\underline{x}(q_t-), \overline{x}(q_t-)) \) it is never optimal for the monopolist to sell to all remaining high types immediately: by doing this the monopolist would charge all remaining high types a price \( P(x_t, \alpha) \), and her profits would be \( g(x_t, q_t-) < L(x_t, q_t-) \). Instead, when costs are in this region the monopolist sells to high types gradually over time, and market penetration increases continuously.\(^{10}\)

The proof of Theorem 1 shows that the seller’s profit function satisfies the following Bellman equation at all states \((x_t, q_t)\) with \( x_t \in (\underline{x}(q_t), \overline{x}(q_t))\):\(^{11}\)

\[
r L(x_t, q_t) = (P(x_t, q_t) - x_t + L_q(x_t, q_t)) \frac{dq_t}{dt} + \mu x_t L_x(x_t, q_t) + \frac{\sigma^2 x_t^2}{2} L_{xx}(x_t, q_t). \tag{12}
\]

The left-hand side of (12) is the seller’s flow payoff at state \((x_t, q_t)\), while the right-hand side shows the sources of this flow payoff. The term \((P(x_t, q_t) - x_t) \frac{dq_t}{dt}\) represents the flow payoff that the seller gets from her sales, while the term \(L_q(x_t, q_t) \frac{dq_t}{dt}\) represents the drop in the seller’s continuation payoff due to the fact that buyers are leaving the market at rate \( dq_t/dt \). Finally, the term last two terms give the change in the seller’s payoff due to changes in costs.

Lemma 3, on the other hand, shows that \( L(x, q) \) satisfies equation (11) for all \( x \in (\underline{x}(q), \overline{x}(q)) \). Comparing equations (12) and (11) it follows that

\[
P(x_t, q_t) - x_t = -L_q(x_t, q_t), \tag{13}
\]

for all \((x_t, q_t)\) such that \( x_t \in (\underline{x}(q_t), \overline{x}(q_t)) \). That is, the profit margin \( P(x_t, q_t) - x_t \) that the monopolist earns is equal to the cost \(-L_q(x_t, q_t)\) that she incurs in terms of a lower continuation payoff. Equation (13) has the following interpretation. The monopolist sells at rate \( dq_t/dt > 0 \) when \( x_t \in (\underline{x}(q_t), \overline{x}(q_t)) \). If \( P(x_t, q_t) - x_t > -L_q(x_t, q_t) \), the monopolist could increase her profits by selling at a faster rate. On the other hand, if \( P(x_t, q_t) - x_t < -L_q(x_t, q_t) \) she would be better off not selling at all. Therefore, for \( dq_t/dt > 0 \) to be optimal, equation (13) must hold for all \( x_t \in (\underline{x}(q_t), \overline{x}(q_t)) \). Figure 3 plots the price \( P(x, q) \) that the

\(^{10}\)In Lemma 3 I derived the lower bound to profits \( L(x, q) \) assuming that the monopolist sells to all high type buyers at the same time. Theorem 1 shows that, in equilibrium, the monopolist obtains profits equal to this lower bound \( L(x, q) \), but through a different strategy. In particular, in equilibrium the monopolist sells to high types gradually over time when \( x_t \in (\underline{x}(q_t-), \overline{x}(q_t-)) \), while in the solution to the optimal stopping problem (10) there are no sales when costs are in this region.

\(^{11}\)Fuchs and Skrzypacz (2010) derive a Bellman equation similar to (12) for their bargaining game with arrival of new buyers.
Figure 3: Prices $P(x,q)$ that the seller charges for $x \in (\bar{x}(q), \bar{x}(q))$; $v_1 = \frac{1}{2}$, $v_2 = 1$, $\alpha = 0.7$, $\mu = -0.02$, $\sigma = 0.25$ and $r = 0.05$.

seller charges when $x \in (\bar{x}(q), \bar{x}(q))$, for different values of $q$.

Note next that, in equilibrium, all high types must get the same payoff; otherwise, it would be profitable for a buyer getting a lower payoff to mimic the strategy of one who is getting a larger payoff. Since high type consumers buy gradually over time when $x \in (\bar{x}(q), \bar{x}(q))$, the evolution of prices must be such that high types are indifferent between buying now or waiting. The proof of Theorem 1 shows that there is a unique rate $dq_t/dt$ at which the monopolist must sell to high type buyers when $x \in (\bar{x}(q), \bar{x}(q))$ in order to maintain their indifference. This rate is given by

$$\frac{dq_t}{dt} = \frac{r(v_2 - x_t) + \mu x_t}{L_{qq}(x_t, q_t)} > 0,$$

where the inequality follows since $L_{qq}(x, q) > 0$ for all $x \in (\bar{x}(q), \bar{x}(q))$ (Lemma B6 in the supplementary appendix) and $r(v_2 - x) + \mu x > 0$ for $x < z_2$.

### 4.2 Features of the equilibrium

In this section I present the most salient features of the equilibrium. I start by discussing how the equilibrium outcome relates to the results on the Coase conjecture. Recall that in markets with time-invariant costs, the Coase conjecture predicts that the monopolist will post an opening price equal to the lowest valuation, and that all buyers will trade immediately at this price. The market outcome will therefore be efficient, and the seller will earn the same profits she would have earned if all buyers in the market had the lowest valuation.
With time-varying costs, selling to all consumers immediately is in general not efficient. By Proposition 1, efficiency requires the monopolist to serve the different types of consumers sequentially as costs decrease. On the other hand, the profits that a monopolist would earn if all consumers had the lowest valuation are \( V_1(x_0) = \sup_{\tau} E[e^{-r\tau} (v_1 - x_\tau)|x_0], \) since in this case she would sell to all buyers at a price equal to \( v_1. \) This observation suggests the following generalization of the Coase conjecture for markets with time-varying costs:

**Definition 2** An outcome is Coasian if (i) it is efficient, and (ii) the monopolist’s profits are equal to \( V_1(x_0). \)

By Definition 2, an outcome is Coasian if it is efficient and if the monopolist earns the same profits she would earn in a market in which all consumers have the lowest valuation. Therefore, under a Coasian outcome the monopolist is unable to extract more rents from buyers with higher valuations than what she extracts from buyers with valuation \( v_1. \) Note that \( V_1(x_0) \to 0 \) as \( v_1 \to 0: \) under a Coasian outcome the seller’s profits converge to zero as the lowest valuation goes to zero.

The equilibrium outcome fails to be Coasian when there are two types of consumers in the market. First, the monopolist is able to extract additional rents from those buyers with higher valuation, since time-varying costs endogenously provide commitment power. Indeed, by Lemma 2 \( P(x, \alpha) - x > V_1(x) \) for all \( x \in (z_1, z_2], \) so \( L(x, 0) \geq \alpha(P(x, \alpha) - x) + (1 - \alpha)V_1(x) > V_1(x) \) for all \( x \in (z_1, z_2]. \) Second, the equilibrium outcome is inefficient whenever \( x_0 \in (x(0), \overline{x}(0)): \) the monopolist sells to high type consumers gradually when costs initially lie within this range, but the efficient outcome is to serve them immediately. Note however that the equilibrium is efficient when costs initially lie either above \( \overline{x}(0) \) or below \( \underline{x}(0). \)

A way to measure the size of the rents that the monopolist extracts from high type buyers is to compare the monopolist’s profits \( L(x, 0) \) to the profits \( \Pi^{FC}(x) \) she would earn if she could commit to a path of prices. In the supplementary appendix, I show that \( \Pi^{FC}(x) = \sup_{\tau} E[e^{-r\tau} \alpha (v_2 - x_\tau)|x_0 = x] \) when \( \alpha v_2 > v_1. \) That is, under full commitment the monopolist would find it optimal to sell only to high types (at a price of \( v_2) \) when the share of high types is large. By Lemma 1, in this case the monopolist would serve high types the first time costs fall below \( z_2. \) High types would be willing to pay a price of \( v_2, \) since the monopolist can commit to keep prices above \( v_2 \) after they purchase. Figure 4 shows that the monopolist may obtain a substantial fraction of the full commitment profits when

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12 The supplementary appendix shows that when \( \alpha v_2 < v_1, \) the monopolist’s full commitment strategy involves selling to high types the first time costs fall below \( z_2, \) and selling to low types the first time costs fall below some \( z < z_1. \)
costs are time-varying.

Another salient feature of the equilibrium is that the price that the monopolist charges at time \( t > 0 \) may depend upon the history of costs. To see this, suppose that \( x_0 \in (\bar{x}(0), \overline{x}(0)) \) and let \( \tau = \inf \{ t : x_t \notin (\bar{x}(q_t), \overline{x}(q_t)) \} \). By equation (14), the rate at which the monopolist sells at time \( s \in [0, \tau) \) depends on the current cost \( x_s \) and on the current level of market penetration \( q_s \). Therefore, for all \( t \in [0, \tau) \) the level of market penetration \( q_t = \int_0^t dq_s \) depends upon the path of costs from time zero to \( t \); and so the price \( P(x_t, q_t) \) that the seller charges at time \( t \) also depends upon the history of costs. Figure 5 plots the path of prices and the evolution of market penetration for a path of costs with \( x_0 \in (\bar{x}(0), \overline{x}(0)) \). Note that \( x_\tau = \overline{x}(q_\tau) \) under this path of costs. Therefore, at time \( \tau \) the monopolist sells to all remaining high type buyers at price \( P(x_\tau, \alpha) \). Since \( P(x, q) \) is increasing in \( x \), at time \( \tau \) a mass of consumers buys at a moment at which prices are increasing.

In settings in which costs are time-invariant, the literature on the Coase conjecture refers to the difference between the lowest consumer valuation and the monopolist’s cost as the gap. When costs don’t change over time, the price that the seller charges to high types is increasing in the gap (since the monopolist’s opening price is equal to the lowest valuation). In this paper’s setting, we can think of the lowest valuation \( v_1 \) as measuring the “gap”. The next result shows that the price \( P(x, \alpha) \) may be increasing or decreasing in the gap when costs are time-varying.

**Proposition 2** For all \( x > z_1 \), \( P(x, \alpha) \) is increasing in \( v_1 \) if and only if \( v_1 \geq \frac{-\lambda}{1-\lambda} v_2 = z_2 \). Moreover, \( \lim_{v_1 \to 0} P(x, \alpha) = v_2 \) for all \( x > 0 \).
The first part of Proposition 2 follows from differentiating $P(x, \alpha)$ in equation (9) with respect to $v_1$. The price at which all high type buyers are willing to buy is decreasing in $v_1$ for low values of $v_1$, and its increasing in $v_1$ for high values of $v_1$. The price $P(x, \alpha)$ depends on two quantities: the time $\tau_1$ at which the monopolist sells to low type consumers (i.e., the endogenous commitment power of the monopolist), and the price $v_1$ that the monopolist charges at time $\tau_1$. An increase in $v_1$ affects both quantities: it decreases the stopping time $\tau_1$ (thereby decreasing $P(x, \alpha)$) and it increases the price $v_1$ that the monopolist charges at $\tau_1$ (thereby increasing $P(x, \alpha)$). By Proposition 2, the second effect dominates when $v_1$ is large, while the first effect dominates when $v_1$ is small.

The second statement in Proposition 2 follows from equation (9) (using the fact that $\lim_{v_1 \to 0} z_1 = 0$). Intuitively, the expected time until costs fall below $z_1$ goes to infinity as $v_1 \to 0$. Hence, as $v_1 \to 0$ the monopolist can commit not to sell to low types for an arbitrarily long period of time after serving high types. Recall that the seller’s full commitment profits are $\Pi^{FC}(x) = \sup_\tau E[e^{-r\tau} \alpha (v_2 - x_\tau) | x_0 = x]$ when $v_1 < \alpha v_2$. Moreover, the seller’s equilibrium profits $L(x, 0)$ are larger than $\sup_\tau E[e^{-r\tau} \alpha (P(x, \alpha) - x_\tau) | x_0 = x]$. Therefore, when $v_1 < \alpha v_2$

$$\sup_\tau E[e^{-r\tau} \alpha (v_2 - x_\tau) | x_0 = x] \geq L(x, 0) \geq \sup_\tau E[e^{-r\tau} \alpha (P(x, \alpha) - x_\tau) | x_0 = x].$$

Since $\lim_{v_1 \to 0} P(x, \alpha) = v_2$, it follows that the seller’s equilibrium profits converge to her full
commitment profits as the gap goes to zero: as \( v_1 \) becomes arbitrarily small the monopolist effectively commits not to sell to low types, since the expected time until costs fall below \( z_1 \) becomes infinitely large.

The next result studies settings in which costs fall deterministically over time by analyzing the limiting properties of the equilibrium when \( \mu < 0 \) and \( \sigma \to 0 \). Note that \( \lim_{\sigma \to 0} \lambda = \frac{z}{\mu} \) and \( \lim_{\sigma \to 0} z_k = \frac{r}{r-\mu} v_k \) when \( \mu < 0 \), so in this limiting case the first best outcome is for the monopolist to serve consumers with valuation \( v_k \) the first time costs fall below \( \frac{r}{r-\mu} v_k \).

**Proposition 3** Suppose \( \mu < 0 \). Then, in the limit as \( \sigma \to 0 \) the monopolist sells to all buyers with valuation \( v_2 \) at the first time costs fall below \( \frac{r}{r-\mu} v_2 \) and sells to all buyers with valuation \( v_1 \) the first time costs fall below \( \frac{r}{r-\mu} v_1 \).

**Proof.** Supplementary appendix. \( \blacksquare \)

When costs fall deterministically over time, the monopolist always sells to buyers with valuation \( v_k \) the first time costs fall below \( \lim_{\sigma \to 0} z_k = \frac{r}{r-\mu} v_k \). That is, in this limiting case the region \((\underline{x}(q), \overline{x}(q))\) in which there is inefficient delay disappears, and the equilibrium outcome becomes efficient. The intuition behind this result is as follows. When \( \sigma > 0 \), there is a chance that costs will go up if the monopolist delays trade with high type buyers. Since \( P(x, \alpha) \) is increasing in \( x \), this increase in costs would allow the monopolist to charge a higher price to consumers with high valuation. This gives rise to an option value of delaying trade. Indeed, Lemma 3 shows that there are cutoffs \( \underline{x}(q) \) and \( \overline{x}(q) > \underline{x}(q) \) such that the monopolist strictly prefers to delay trade with high types than to sell to all of them immediately when costs are in \((\underline{x}(q), \overline{x}(q))\). However, the probability that costs will increase becomes negligible when \( \mu < 0 \) and \( \sigma \to 0 \). Therefore, in this limiting case it is no longer profitable to delay trade with high type consumers when costs are below \( \frac{r}{r-\mu} v_2 \), and the market outcome becomes efficient.

Although the equilibrium is efficient when \( \mu < 0 \) and \( \sigma \to 0 \), the monopolist is still able to extract rents from high type buyers in this setting in which costs fall deterministically over time: when costs are above \( \lim_{\sigma \to 0} z_1 = \frac{r}{r-\mu} v_1 \), high type buyers know that it will take a non-negligible amount of time for prices to fall to \( v_1 \), so they are willing to pay a higher price. Therefore, the equilibrium outcome is not Coasian in this setting.

The next result studies the limiting properties of the equilibrium as the drift and volatility of costs converge to zero; i.e., as costs become time-invariant. For any \( x \in [0, v_2] \), let \( p(x) \) denote the lowest consumer valuation that is weakly larger than \( x \); that is, \( p(x) = v_1 \) if \( x \leq v_1 \), while \( p(x) = v_2 \) if \( x \in (v_1, v_2] \).
**Proposition 4** Suppose \( x_0 \leq v_2 \). Then, as \((\sigma, \mu) \to (0, 0)\) the monopolist sells at \( t = 0 \) to all consumers with valuation larger than \( x_0 \) at a price \( p(x_0) \).

**Proof.** Supplementary appendix. □

Proposition 4 shows that the equilibrium outcome of this model converges to the standard Coase conjecture outcome when costs become time-invariant. For instance, if costs are initially below \( v_1 \), the monopolist’s opening price converges to the lowest valuation \( v_1 \) as \((\sigma, \mu) \to (0, 0)\), and all consumers trade immediately at this price.

## 5 Markets with \( n > 2 \) types of consumers

In this section I show how the results in Section 4 generalize to settings in which the function \( f : [0, 1] \to [\underline{v}, \overline{v}] \) describing the valuations of the consumers takes \( n \) values \( v_1 < \ldots < v_n \), with \( n > 2 \). For \( k = 1, \ldots, n \), recall that \( \alpha_k = \max \{ i \in [0, 1] : f(i) = v_k \} \) is the highest indexed consumer with valuation \( v_k \), and let \( \alpha_{n+1} = 0 \). Note then that \( f(i) = v_k \) for all \( i \in (\alpha_{k+1}, \alpha_k] \).

As a first step, note that at states \((x, q)\) with \( q \in [\alpha_3, 1) \) there are one or two types of buyers in the market: buyers with valuation \( v_1 \) and, if \( q < \alpha_2 \), buyers with valuation \( v_2 \). Thus, for states \((x, q)\) with \( q \in [\alpha_3, 1) \) the equilibrium is the one derived in Section 4: for \( q \geq \alpha_2 \) the seller’s profits are \((1 - q)V_1(x)\), and for \( q \in [\alpha_3, \alpha_2) \) the seller’s profits are equal to the lower bound \( L(x, q) = \sup_{\tau} \mathbb{E}[e^{-r\tau}((\alpha_2 - q)(P(x_\tau, \alpha_2) - x_\tau) + (1 - q)V_1(x))|x_0 = x] \), where \( P(x_\tau, \alpha_2) = v_2 - E[e^{-r\tau_2}(v_2 - v_1)|x_0 = x] \) is the price at which the monopolist can sell to all buyers with valuation \( v_2 \).

Consider next states \((x, q)\) with \( q \in [\alpha_4, \alpha_3) \). At such a state there are \( \alpha_3 - q \) buyers with valuation \( v_3 \) remaining in the market. By equation (5), the strategy \( P(x, \alpha_3) \) of consumer \( \alpha_3 \) (the highest indexed buyer with valuation \( v_3 \)) satisfies \( P(x, \alpha_3) = v_3 - E[e^{-r\tau_2}(v_3 - P_2(x_{\tau_2}, q_{\tau_2}))|x_0 = x] \), where \( \tau_2 = \inf\{ t : x_t \leq z_2 \} \) is the time at which the monopolist starts selling to consumers with valuation \( v_2 \) when all consumers with valuation \( v_3 \) have left the market. The skimming property implies that \( P(x, i) \geq P(x, \alpha_3) \) for all \( i \leq \alpha_3 \), so the monopolist can sell to all buyers with valuation \( v_3 \) at price \( P(x, \alpha_3) \). Therefore, at states \((x, q)\) with \( q \in [\alpha_4, \alpha_3) \) the seller’s profits are bounded below by

\[
L(x, q) = \sup_{\tau \in \mathcal{T}} \mathbb{E} \left[ e^{-r\tau} \left( (\alpha_3 - q)(P(x_\tau, \alpha_3) - x_\tau) + L(x_\tau, \alpha_3) \right) \right] \big| x_0 = x, \quad (15)
\]

where \( L(x, \alpha_3) \) are the seller’s profits at state \((x, \alpha_3)\). By arguments similar to those in Lemma 3, the solution to (15) involves delaying when costs are either in an interval around
or in an interval around $z_2$: in these regions, the seller obtains a larger payoff by waiting than by selling immediately to all buyers with valuation $v_3$ at price $P(x, \alpha_3)$. The solution to (15) also involves delaying when $x > z_3$, since producing at such cost levels is too expensive relative to the valuation $v_3$. As in the two types case, the seller’s profits are equal to $L(x, q)$ for all states $(x, q)$ with $q \in [\alpha_4, \alpha_3)$ (see Theorem 2 below). When $x_t < z_3$ is in the delay region of (15), the monopolist sells gradually to buyers with valuation $v_3$. When $x_t$ is in the stopping region of (15), the monopolist sells to all remaining buyers with valuation $v_3$ at price $P(x_t, \alpha_3)$. When $x_t > z_3$ the monopolist doesn’t sell.

Following the same steps as in the derivation of (15), I can extend $L(x, q)$ to all $q \in [0, 1]$ in such a way that, for $k = 2, ..., n$ and all $q \in [\alpha_{k+1}, \alpha_k)$,

$$L(x, q) = \sup_{\tau \in T} E \left[ e^{-r\tau} \left( (\alpha_k - q) (P(x, \alpha_k) - x) + L(x, \alpha_k) \right) \right]_{x_0 = x}, \quad (16)$$

where $P(x, \alpha_k)$ is the price consumer $\alpha_k$ is willing to pay, and $L(x, \alpha_k)$ is the lower bound to the monopolist’s profits at state $(x, \alpha_k)$. For $q \in [\alpha_2, 1]$, let $L(x, q) = (1 - q)V_1(x)$. Theorem 2 below generalizes Theorem 1 to the case with any finite number of consumer types.

**Theorem 2** Suppose $f$ is a step function taking $n > 2$ values. Then, there exists a unique equilibrium. In equilibrium, the monopolist’s profits are equal to $L(x, q)$ at every state $(x, q)$.

**Proof.** Supplementary appendix. ■

Theorem 2 shows the seller’s equilibrium profits are also equal to the lower bound $L(x, q)$ in this more general setting. At states $(x, q)$ with $q \in [\alpha_{k+1}, \alpha_k)$, buyers with valuation $v_{k+1}$ and higher have already left the market. At these states, the solution to (16) involves delaying when $x$ is in an interval around $z_1$, $z_2$, ..., or $z_{k-1}$, and when $x > z_k$. When $x < z_k$ is in the delay region of the optimal stopping problem (16) the monopolist sells gradually to buyers with valuation $v_k$ (the highest valuation remaining in the market) at price $P(x, q) = x - L_q(x, q)$. If $x > z_k$, the seller waits until costs fall to $z_k$, and at this point sells to all buyers with valuation $v_k$ at price $P(x, \alpha_k)$. Finally, if $x$ lies in the stopping region of (16), the seller serves all remaining buyers with valuation $v_k$ at price $P(x, \alpha_k)$.

The market outcome also fails to be Coasian in this setting. First, the monopolist is able to extract additional rents from those buyers with higher valuations. Indeed, arguments similar to those in Lemma 2 imply that, for all $k \geq 2$, $P(x, \alpha_k) - x > V_1(x)$ for all $x \in (z_1, z_k]$. Since the monopolist can sell to all consumers with valuation $v_k$ and higher at a price of $P(x, \alpha_k)$, it follows that $L(x, q) > (1 - q)V_1(x)$ for all $x \in (z_1, z_k]$ and $q < \alpha_2$. Intuitively, after selling
to buyers with valuation $v_k$ the seller can commit to keep prices high until costs fall below $z_{k-1}$; and this commitment power allows her to extract rents from consumers with valuations $v_k$ and higher. Second, there is inefficient delay in equilibrium: when $x_0 < z_n$ lies in the delay region of (16), the efficient outcome is to serve all buyers with valuation $v_n$ immediately, but the monopolist serves them gradually.

6 Markets with a continuum of types

In this section I study markets in which the buyers’ valuations are described by a continuous and decreasing function $h: [0, 1] \to \mathbb{R}_+$, with $h(0) = \bar{v} > v = h(1) > 0$. I study such markets by considering a sequence of models $j = 1, 2, \ldots$ with functions $\{f^j\}$, with $f^j : [0, 1] \to [\bar{v}, v]$ taking finitely many values for all $j$, such that $\{f^j\} \to h$ (i.e., $\sup_{i \in [0,1]} |f^j(i) - h(i)| \to 0$ as $j \to \infty$).\(^{13}\)

Given such a sequence $\{f^j\}$, let $L^j(x, q)$ denote the monopolist’s profits at state $(x, q)$ in an environment in which the valuations of the consumers are described by $f^j$. Let $V(x) = \sup_r E[e^{-rt}(\bar{v} - x_t)|x_0 = x]$ be the profits that the seller would earn if all buyers in the market had the lowest valuation $\bar{v} > 0$.

**Theorem 3** Fix a sequence of step functions $\{f^j\}$ such that $\{f^j\} \to h$. Then, the equilibrium outcome becomes Coasian as $j \to \infty$: (i) $\lim_{j \to \infty} L^j(x, 0) = V(x)$ for all $x$, and (ii) in the limit $j \to \infty$ the monopolist serves the different consumers at the efficient time.

**Proof.** Appendix A.2. 

To see intuition behind Theorem 3, consider first a setting with two types of buyers: high types, with valuation $\bar{v}$, and low types, with valuation $v$. After high types buy and leave the market, the monopolist can truthfully commit to keep high prices until costs fall below $z = \frac{-\lambda}{1-\lambda} v$. High type buyers know that prices won’t fall to $v$ until $x_t$ falls below $z$, so they are willing to pay higher prices when costs are above $z$. Consider next a setting with three types of buyers, with valuations $\bar{v}$, $(\bar{v} + v)/2$ and $v$. In this setting, after all consumers with valuation $\bar{v}$ buy and leave the market, the monopolist can only commit not to cut prices until costs fall below $\frac{-\lambda}{1-\lambda} \frac{\bar{v} + v}{2}$, since at this point it becomes optimal for her to sell to buyers with intermediate valuation. This puts a limit to the price buyers with valuation $\bar{v}$ are willing to pay, since they can now wait for costs to fall to $\frac{-\lambda}{1-\lambda} \frac{\bar{v} + v}{2}$ and get the good at a lower price.

\(^{13}\)For all $j$, $f^j$ satisfies the assumptions in Section 2.
More generally, the proof of Theorem 3 shows that the price consumers are willing to pay monotonically decreases as types become a continuum. In the limit as the gap between valuations becomes vanishingly small, the monopolist cannot extract additional rents from buyers with higher valuations, and her profits fall to what she would earn in a market in which all consumers have the lowest valuation $v$. Intuitively, the monopolist loses all commitment power when she faces a continuum of types, since in this case she always has an incentive to serve the next consumer arbitrarily soon after her last sale.

The equilibrium outcome is efficient with a continuum of types: the monopolist serves consumers with valuation $v \in [\underline{v}, \overline{v}]$ the first time costs fall below the threshold $z_v = \frac{\lambda}{1-\lambda} v$. This implies that for all $t \geq 0$, consumers with valuation above $\frac{1-\lambda}{\lambda} \times \min_{s \leq t} x_t$ have bought and left the market. Therefore, with a continuum of types the level of market penetration at each time $t$ is entirely determined by the lowest level of costs up to $t$, and market penetration only increases when costs fall below their minimum historical level.

The proof of Theorem 3 shows that the price that the monopolist charges to consumers with valuation $v$ is $x_{\tau_v} + \mathcal{V}(x_{\tau_v})$, where $\tau_v = \inf \{ t : x_t \leq \frac{\lambda}{1-\lambda} v \}$ is the time at which these consumers make their purchase. Let $H$ denote the cumulative distribution function of valuations implied by $h : [0, 1] \to [\underline{v}, \overline{v}]$. Since the monopolist sells to buyers with valuation $v$ at time $\tau_v$ and obtains a profit margin of $\mathcal{V}(x_{\tau_v})$, her profits are

$$E \left[ \int_{\underline{v}}^{\overline{v}} e^{-r \tau_v} \mathcal{V}(x_{\tau_v}) \, dH(v) \bigg| x_0 \right] = \int_{\underline{v}}^{\overline{v}} E \left[ e^{-r \tau_v} \mathcal{V}(x_{\tau_v}) \bigg| x_0 \right] \, dH(v) = \mathcal{V}(x_0),$$

where the last equality follows since $E[e^{-r \tau_v} \mathcal{V}(x_{\tau_v}) \big| x] = \mathcal{V}(x)$ for all $x$ and all $v \in [\underline{v}, \overline{v}]$.

The price $x + \mathcal{V}(x)$ that the monopolist charges in the limiting equilibrium is increasing $x$. Since consumers with higher valuations buy earlier, at higher levels of costs, they pay higher prices. However, these higher prices are offset by the higher levels of costs at which consumers with higher valuations make their purchase: the profit margin that the monopolist earns on higher valuation buyers is exactly equal to the expected discounted profit margin that she earns on consumers with valuation $v$.

The assumption that the function $h : [0, 1] \to [\underline{v}, \overline{v}]$ is continuous implies that the distribution of consumer valuations has convex support. This assumption is crucial for Theorem 3. To see this, suppose that the function $h$ has a single discontinuity point at $j \in (0, 1)$, with $h(j) > h(j^+) = \lim_{i \uparrow j} h(i)$. In such a setting, after selling to all consumers $i \leq j$
the monopolist can commit not to reduce her price until costs fall below \( \frac{-\lambda}{1-\lambda} h(j^+) \). This commitment ability allows the seller to extract rents from buyers with valuation larger than \( h(j) \) when costs are above \( \frac{-\lambda}{1-\lambda} h(j^+) \). Moreover, this ability of the monopolist to extract rents creates a wedge between profit maximization and efficiency, just as in the two types case of Section 4. As a result, the market outcome fails to be efficient in this setting.

One possible interpretation of a setting in which the distribution of valuations has non-convex support is that it represents a market in which there is a clear segmentation among different groups of consumers. Examples of such segmented markets may include a monopolist selling at different geographic locations, or in the case of intermediate durable goods, a monopolist selling to firms in different industries. The results in this paper then suggest that a durable goods monopolist with time-varying costs will be able to obtain more profits in such segmented markets.

I now turn to study settings in which costs fall deterministically over time. I do this by analyzing the limiting properties of the market outcome when \( \mu < 0 \) and \( \sigma \to 0 \). Note that \( \lim_{\sigma \to 0} \lambda = r/\mu \) when \( \mu < 0 \). This implies that \( z_v = \frac{-\lambda}{1-\lambda} v \to z^*_v = \frac{r}{r-\mu} v \) as \( \sigma \to 0 \), so in the limit the monopolist sells to consumers with valuation \( v \) the first time costs fall below \( z^*_v = \frac{r}{r-\mu} v \). Moreover, from equation (7) it follows that \( \lim_{\sigma \to 0} V(x) = (v - z^*_v) \left( x/z^*_v \right)^{r/\mu} \) for all \( x > z^*_v = \frac{r}{r-\mu} v \). Therefore, the monopolist’s profits converge to \( (v - z^*_v) \left( x/z^*_v \right)^{r/\mu} \) as \( \sigma \to 0 \). The following Proposition summarizes these results.

Proposition 5 Suppose \( \mu < 0 \) and \( x_0 > v \). Then, as \( \sigma \to 0 \) the monopolist’s profits converge to \( (v - z^*_v) \left( x_0/z^*_v \right)^{r/\mu} \). Moreover, in the limit the monopolist serves consumers with valuation \( v \) the first time costs fall below \( z^*_v = \frac{r}{r-\mu} v \).

As noted in Section 4.2, in markets with time-varying costs we can think of the lowest valuation \( v > 0 \) as measuring the “gap”. The final result of this section studies the limiting properties of the market outcome as the gap goes to zero.

Proposition 6 As \( v \to 0 \), the monopolist charges price equal to marginal cost.

Proposition 6 follows from the characterization of the market outcome with a continuum of types. To see this, note that \( V(x) = (v - z^*_v) \left( x/z^*_v \right)^{\lambda} \to 0 \) as \( v \to 0 \). Therefore, the market outcome converges to the perfectly competitive outcome as the gap goes to zero: in the limit as \( v \to 0 \) the monopolist sells and a price equal to marginal cost and earns zero profits.
7 Conclusion

In this paper, I study the problem of a durable goods monopolist who lacks commitment power and who faces uncertain and time-varying costs. At a general level, the model in this paper is one of supply shocks. This contrasts with previous papers in the literature that studied how demand shocks affect durable goods markets.\textsuperscript{16} I show that a durable goods monopolist with time-varying costs usually serves the different types of consumers at different points in time and charges them different prices. The market outcome in this setting fails to be Coasian when the distribution of valuations has non-convex support, since time-varying costs endogenously provide commitment power to the monopolist. On the other hand, with a continuum of types the market outcome is Coasian: the monopolist is unable to extract rents, and the market outcome is efficient.

A Appendix

A.1 Proofs of Theorem 1

Proof of Theorem 1. The supplementary appendix B.3 shows that, in any equilibrium, the monopolist’s profits are equal to $L(x, q)$ for all $(x, q)$ with $q < \alpha$. Here, I complete the proof of Theorem 1 by constructing the unique equilibrium. In order for the monopolist to obtain profits equal to $L(x, q)$, she must sell to all high types at price $P(x_t, \alpha)$ when $x_t \in [0, \underline{x}(q_t)] \cup [\bar{x}(q_t), z_2]$ (and also to low types when $x_t \leq \underline{x}(q_t)$). Moreover, by Lemma B7 in the supplementary appendix, the monopolist sells at a positive rate when $x_t \in (\underline{x}(q_t), \bar{x}(q_t))$ (i.e., $dq_t > 0$). I now determine the price that the monopolist charges and the rate at which she sells when $x_t \in (\underline{x}(q_t), \bar{x}(q_t))$. Suppose $x_t \in (\underline{x}(q_t), \bar{x}(q_t))$ and let $\tau = \inf\{s > t : x_s \notin (\underline{x}(q_s), \bar{x}(q_s))\}$. At any $s \in [t, \tau)$, the seller’s expected discounted profits (which are equal to $L(x_s, q_s)$) are given by

$$L(x_s, q_s) = E\left[\int_s^\tau e^{-r(u-s)} (P(x_u, q_u) - x_u) \, dq_u + e^{-r(\tau-s)} L(x_\tau, q_\tau) \bigg| F_s\right].$$

By the Law of Iterated Expectations, the process

$$Y_s = \int_0^s e^{-ru} (P(x_u, q_u) - x_u) \, dq_u + e^{-rs} L(x_s, q_s) \quad \text{(A.1)}$$

\textsuperscript{16}Examples of models with demand shocks include settings with arrivals of new buyers (Sobel, 1991) and settings with stochastic shocks to valuations (Biehl, 2001).
Equation (A.3) shows that (in expectation) prices must fall at rate \( P \) to keep high types indifferent between buying at any time \( s\) with 
\[
E \left[ v_2 - P(x, q) \right] = 0.
\]
Since \( L(x, q) \in C^2 \) for all \( x \in (\underline{x}(q), \overline{x}(q)) \) (Lemma B5 in Appendix B.2), by Ito’s Lemma
\[
dL(x, q) = \left( \mu x L(x, q) + \frac{\sigma^2 x^2}{2} L_{xx}(x, q) \right) dt + \sigma x L_x(x, q) dW_t.
\]
Combining these two equations, the seller’s profit function satisfies
\[
rL(x, q) ds = (P(x, q) - x) dq + \mu x L(x, q) ds + \frac{\sigma^2 x^2}{2} L_{xx}(x, q) ds,
\]
at all states \((x, q)\) with \( x \in (\underline{x}(q), \overline{x}(q))\). On the other hand, the proof of Lemma 3 shows that \( L(x, q) \) satisfies equation (11) for all \( x \in (\underline{x}(q), \overline{x}(q))\). Comparing equation (11) with the equation above, it follows that \( P(x, q) - x = -L_q(x, q) \) for all \((x, q)\) with \( x \in (\underline{x}(q), \overline{x}(q))\). This expression pins down the prices that the seller charges when \( x \in (\underline{x}(q), \overline{x}(q)) \) (and hence the buyers’ strategy profile in this region).

Finally, I pin down the rate \( dq \) at which the monopolist sells when \( x_t \in (\underline{x}(q_t), \overline{x}(q_t)) \). Note that all high types must get the same payoff; otherwise, it would be profitable for a buyer getting a lower payoff to mimic the strategy of one who is getting a larger payoff. Therefore, prices must evolve in such a way that high types are indifferent between buying at any \( s \in [t, \tau) \) (where \( \tau = \inf \{ s > t: x_s \notin (\underline{x}(q_s), \overline{x}(q_s)) \} \). That is,
\[
e^{-rs} (v_2 - P(x, q)) = E \left[ e^{-ru} (v_2 - P(x, q)) \right] | F_s], 
\]
for any \( s, u \in [t, \tau) \), \( s < u \). By the Law of Iterated Expectations, the process \( M_s = E \left[ e^{-ru} (v_2 - P(x, q)) \right] | F_s \) is a continuous martingale. By the Martingale Representation Theorem, there exists a progressively measurable process \( \gamma \) such that \( dM_s = e^{-rs} \gamma_s dB_s \). Differentiating (A.2) with respect to \( s \) gives
\[
dM_s = -re^{-rs} (v_2 - P(x, q)) ds - e^{-rs} dP(x, q) \Rightarrow 
\]
\[
dP(x, q) = -r (v_2 - P(x, q)) ds - \gamma_s dB_s.
\]
Equation (A.3) shows that (in expectation) prices must fall at rate \( -r (v_2 - P(x, q)) \) in order to keep high types indifferent between buying at any time \( s \in [t, \tau) \). By the arguments above, \( P(x, q) = x - L_q(x, q) \) for all \( s \in [t, \tau) \). The proof of Lemma 3 shows that \( L(x, q) \in C^2 \) for all \( x \in (\underline{x}(q), \overline{x}(q)) \), so \( P(x, q) \in C^{2,1} \) for all \( x \in (\underline{x}(q), \overline{x}(q)) \). Ito’s Lemma then gives
\[
dP(x, q) = \left( \mu x P_x(x, q) + \frac{\sigma^2 x^2}{2} P_{xx}(x, q) \right) ds + P_q(x, q) dq + P_x(x, q) \sigma x dB_s,
\]
for all $s \in [t, \tau)$. This equation and (A.3) give two expressions for $dP(x_s, q_s)$. Since these expressions must be equal, it follows that $\frac{dq}{ds} = (-r(v_2 - P(x_s, q_s)) - \mu x P_x(x_s, q_s) - \frac{1}{2} \sigma^2 v_2^2 P_{xx}(x_s, q_s))/P_q(x_s, q_s)$. The proof of Lemma 3 shows that $L_q(x, q)$ solves $rL_q(x, q) = \mu x L_{qq}(x, q) + \frac{\sigma^2 x^2}{2} L_{qxx}(x, q)$ for all $x \in (x(q), \bar{x}(q))$ (see footnote 1 in Appendix B.2). Using this with the fact that $P(x_s, q_s) - x_s = -L_q(x_s, q_s)$ gives $\frac{dq}{ds} = -r(v_2 - x_s) + \mu x = \frac{r(v_2 - x_s) + \mu x}{L_{qq}(x_s, q_s)} > 0$, where the equality follows since $P_q(x, q) = -L_{qq}(x, q)$, and the inequality follows since $L_{qq}(x, q) > 0$ for all $x \in (x(q), \bar{x}(q))$ (Lemma B6 in Appendix B.2) and since $r(v_2 - x) + \mu x > 0$ for all $x < z_2$.

### A.2 Proof of Theorem 3

Fix a sequence $\{f^j\} \to h$, with $f^j : [0, 1] \to [v, \bar{v}]$ taking finitely many values for all $j$. For $j = 1, 2, ..., \eta$, let $v^j \downarrow v^j_1 < v^j_2 < ... < v^j_n$ be the set of possible valuations under $f^j$. For $k = 1, ..., n$, let $z_k^j = \frac{v^j_k - v^j_{k-1}}{\lambda}$. For each $j$, define the function $P^j(x)$ as follows. For $x \leq z_k^j$, $P^j(x) = v^j_k$. For $k = 2, ..., n$, and $x \in (z_k^j, z_k^{j+1}]$, $P^j(x) = P^j(x, \alpha_k^j)$. Equation (5) then implies that, for $k = 2, ..., n$, and $x \in (z_k^j, z_k^{j+1}]$, $P^j(x) = v^j_k - E[e^{-r(t-x)}(v^j_k - P^j(z_k^{j-1}))|x_0 = x]$, where for $k = 1, ..., n$, $\tau_j = \inf \{t : x_t \leq \lambda\}$ is the time at which the monopolist starts selling to buyers with valuation $v^j_k$ when $v^j_k$ is the highest valuation remaining in the market.

**Lemma A1** For $k = 2, ..., n$, and $x \in (z_k^{j-1}, z_k^j]$, $P^j(x) = v^j_k - \sum_{m=1}^{k-1}(v^j_m - v^j_k)(x/z^j_m)\lambda$.

**Proof.** The proof is by induction. By equation (9) in the main text, $P^j(x) = v^j_k - (v^j_k - P^j(z_k^{j-1}))(x/z_k^{j-1})\lambda$ for all $x \in (z_k^{j-1}, z_k^j]$. The induction hypothesis then implies that $P^j(x) = v^j_k - (v^j_k - P^j(z_k^{j-1}))(x/z_k^{j-1})\lambda = v^j_k - \sum_{m=1}^{k-1}(v^j_m - v^j_k)(x/z^j_m)\lambda$ for $x \in (z_k^{j-1}, z_k^j]$. Let $\bar{z} = \frac{-1}{1-\lambda}v$ and $\underline{z} = \frac{-1}{1-\lambda}v$, and let $V(x) = \sup_x E[e^{-r(t-x)}(v - x)|x_0 = x]$. By Lemma 1, $V(x) = E[e^{-r(t-x)}(v - x)|x_0 = x]$, where $\bar{z} = \inf \{t : x_t \leq \bar{z}\}$.

**Lemma A2** $P^j(x) - x \to V(x)$ uniformly on $[0, \bar{z}]$ as $j \to \infty$.

**Proof.** I first show that $\lim_{j \to \infty} P^j(x) = V(x) + x$ for all $x \in [0, \bar{z}]$. Note first that, for all $x \leq \underline{z}$, $\lim_{j \to \infty} P^j(x) = \lim_{j \to \infty} v^j_1 = \bar{v}$, $V(x) + x$. Next, fix $x \in (\bar{z}, \bar{z}]$ and for $j = 1, 2, ..., \eta$, let $k_j$ be such that $x \in (z_{k_j-1}^j, z_{k_j}^j]$. Let $v(x) = \frac{1-\lambda}{\lambda}x$. Lemma A1 and the fact that $x/z^j_m = v(x)/v^j_m$ imply that $P^j(x) = v^j_{k_j} - \sum_{m=1}^{k_j-1}(v^j_m - v^j_{k_j})(v(x)/v^j_m)\lambda$. Since $x \in (z_{k_j-1}^j, z_{k_j}^j]$ for all $j$ and since $\lim_{j \to \infty} z_{k_j-1}^j - z_{k_j}^j = 0$, it follows that $z_{k_j}^j = \frac{-1}{1-\lambda}v^j_{k_j} \to x$ as $j \to \infty$. Hence, $\lim_{j \to \infty} v^j_{k_j} = \frac{1-\lambda}{\lambda}v = v(x)$. Since $(v(x)/v)^\lambda$ is Riemann integrable, $\lim_{j \to \infty} P^j(x) = v(x) - \int_{\bar{z}}^{x}(v(x)/v)^\lambda dv = x + (\bar{v} - \underline{z})(x/\bar{z})\lambda = x + V(x)$. Finally, since $P^j(x)$ is increasing in $x$ for all $j$ and since $\lim_{j \to \infty} P^j(x) = V(x) + x$ for all $x \in [0, \bar{z}]$, it follows that $P^j(x) \to V(x) + x$ uniformly on $[0, \bar{z}]$ as $j \to \infty$. Thus, $P^j(x) - x \to V(x)$ uniformly on $[0, \bar{z}]$ as $j \to \infty$. 


Proof of Theorem 3. I first prove that, for all \( x \), \( L^j(x,0) \rightarrow V(x) \) as \( j \rightarrow \infty \). Note first that \( L^j(x,0) \geq V(x) \) for all \( x \), since at any state \((x,0)\) the monopolist can wait until time \( \tau_1 \) and sell to all buyers at price \( v_1 \geq V \), obtaining a profit of \( E[e^{-rt}v_1^j(x_{\tau_1}^j) \mid x_0=x] \geq V(x) \).

Next, consider the case in which \( x_0 = x \geq \bar{z} \). In this case, in equilibrium the monopolist sells to consumers with valuation \( v_k^j \) at time \( \tau_k^j = \inf\{t : x_t \leq z_k \} \) for \( k = 1, \ldots, n_j \) at a price \( P(z_k^j, \alpha_k^j) = p_j^j(z_k^j) \) (where \( \alpha_k^j = \max\{i : f^j(i) = v_k^j\} \)). Let \( \alpha_{n,j+1}^j = 0 \). Then, the seller’s profits are \( L^j(x,0) = \sum_{k=1}^{n_j} e^{-rt} k P_j^j(z_k^j) \| x_0 = x \| (\alpha_k^j - \alpha_{k+1}^j) \). Since \( P_j^j(x) - x \rightarrow V(x) \) uniformly on \([0, \bar{z}]\) as \( j \rightarrow \infty \), for every \( \eta > 0 \) there exists \( N \) such that \( P_j^j(x) - x \rightarrow V(x) < \eta \) for all \( j > N \) and all \( x \in [0, \bar{z}] \). Thus, for \( j > N \), \( \sum_{k=1}^{n_j} e^{-rt} k (P_j^j(z_k^j) \| x_0 = x \| ) (\alpha_k^j - \alpha_{k+1}^j) < \sum_{k=1}^{n_j} \alpha_k^j \). Note further that for \( x \geq \bar{z} \) and \( k = 1, 2, \ldots, n_j \), \( \sum_{k=1}^{n_j} \alpha_k^j = \alpha_j^j \) (so \( \sum_{k=1}^{n_j} \alpha_k^j = 1 \)). Note further that for \( x \geq \bar{z} \) and \( k = 1, 2, \ldots, n_j \), \( E[e^{-rt} \tau_j^j L_j^j(x_{\tau_j^j}) | x_0 = x] = E[e^{-rt} \tau_j^j E[r(\bar{z}-t\bar{z}) | x_{\tau_j^j}] | x_0 = x] = V(x) \). Using this and the fact \( \sum_{k=1}^{n_j} \alpha_k^j = 1 \), it follows that \( V(x) \leq L^j(x,0) < V(x) + \eta \) for all \( j > N \). Therefore, \( \lim_{j \rightarrow \infty} L^j(x,0) = V(x) \) for all \( x \geq \bar{z} \).

Consider next the case with \( x_0 < \bar{z} \), and suppose by contradiction that \( L^j(x,0) \rightarrow V(x) \) as \( j \rightarrow \infty \). Since \( L^j(x,0) \geq V(x) \) for all \( n \), there exists a subsequence \( \{j_r\} \), \( N > 0 \) such that \( L^{j_r}(x,0) > V(x) + \gamma \) for all \( j_r > N \). Fix \( y \geq \bar{z} \) and let \( \tau_x = \inf\{t : x_t \leq x \} \). Since the seller can delay trade until time \( \tau_x \), it must be that \( L^{j_r}(y,0) \geq E[e^{-rt} L^{j_r}(x_{\tau_x},0) | x_0 = y] \). Thus, for all \( j_r > N \), \( L^{j_r}(y,0) > E[e^{-rt} \tau_x V(x_{\tau_x}) | x_0 = y] + \eta \). But this contradicts \( \lim_{j_r \rightarrow \infty} L^{j_r}(y,0) = V(y) \), since \( E[e^{-rt} \tau_x V(x_{\tau_x}) | x_0 = y] = V(y) \) and since \( E[e^{-rt} \tau_x | x_0 = y] = \gamma(\bar{z}) > 0 \). Thus, \( \lim_{j \rightarrow \infty} L^j(x,0) = V(x) \).

Finally, I show that the limiting equilibrium outcome is efficient. Note that Lemma A2 implies that, for all \( i \in [0, 1] \), the price \( P_i^j(x, i) \) that consumer \( i \) is willing to pay converges to \( V(x) + x \) for all \( x \leq \sum_{k=1}^{n_j} \alpha_k^j \ell(i) \) as \( j \rightarrow \infty \). This in turn implies that, in the limit as \( j \rightarrow \infty \), the monopolist always sells at price \( V(x_{t}) + x_{t} \). By the same arguments used in the proof of Lemma 1, \( \tau_v = \inf\{t : x_t \leq \sum_{k=1}^{n_j} \alpha_k^j \ell(i) \} \) solves \( \sup_{\tau} E[e^{-rt} \gamma (v - V(x_{\tau}) - x_{\tau}) | x_0 = x] \) for all \( v \in [x, \bar{z}] \). Since the seller always charges \( V(x_{t}) + x_{t} \) as \( j \rightarrow \infty \) and since buyers buy at the time that maximizes their surplus, in the limit a buyer with valuation \( v \) buys at time \( \tau_v \).

B Omitted proofs

B.1 Proofs of Lemmas 1 and 2

Fix \( y_2 > y_1 > 0 \) and let \( \tau_y = \inf\{t : x_t \notin (y_1, y_2)\} \). Let \( \tau_{y_1} = \inf\{t : x_t \leq y_1\} \). Note that \( \tau_y \) and \( \tau_{y_1} \) are random variables, whose distributions depend on the initial level of costs \( x_0 \).

Lemma B1 Let \( g \) be a bounded function, and let \( W(x) \) be the solution to

\[
rW(x) = \mu x W'(x) + \frac{1}{2} \sigma^2 x^2 W''(x),
\]

with \( W(y_i) = g(y_i) \) for \( i = 1, 2 \). Then, \( W(x) = E[e^{-rt} g(x_{\tau_y}) | x_0 = x] \) for all \( x \in (y_1, y_2) \).
Proof. Let $W$ satisfy (B.1) with $W(y_1) = g(y_1)$ and $W(y_2) = g(y_2)$. The general solution to (A.1) is $W(x) = Ax^\lambda + Bx^\kappa$, where $\lambda < 0$ and $\kappa > 1$ are the roots of $\frac{1}{2}\sigma^2\lambda (\lambda - 1) + \mu\lambda = r$, and where $A$ and $B$ are constants determined by the boundary conditions:

$$A = \frac{g(y_2) y'_1 - g(y_1) y'_2}{y'_1 y'_2 - y'_1 y'_2} \quad \text{and} \quad B = -\frac{g(y_2) y'_1 - g(y_1) y'_2}{y'_1 y'_2 - y'_1 y'_2} \quad (B.2)$$

Let $f(x, t) = e^{-rt}W(x)$. By Itô’s Lemma, for all $x_t \in (y_1, y_2)$

$$df(x, t) = e^{-rt} \left( -rW(x_t) + \mu xW'(x_t) + \frac{1}{2} \sigma^2 x^2 W''(x_t) \right) dt + e^{-rt} \sigma x W'(x_t) dB_t$$

where the second equality follows from the fact that $W$ solves (B.1). Then,

$$E \left[ e^{-r\tau_y} g(x_{\tau_y}) \mid x_0 = x \right] = E \left[ f(x_{\tau_y}, \tau_y) \mid x_0 = x \right] = f(x, 0) + E \left[ \int_0^\tau df(x, t) \mid x_0 = x \right] = W(x)$$

since $\int_0^\tau e^{-rt} \sigma x W'(x_t) dB_t$ is a Martingale with expectation zero. $lacksquare$

Corollary B1 Let $g$ be a bounded function, and let $w$ be a solution to (B.1) with $w(y_1) = g(y_1)$ and $\lim_{x \to \infty} w(x) = 0$. Then, $w(x) = E[e^{-r\tau_y} g(x_{\tau_y}) \mid x_0 = x]$ for all $x > y_1$. Moreover, $w(x) = g(y_1) (x/y_1)^\lambda$ for all $x > y_1$.

Proof. Since $w$ solves (B.1), it follows that $w(x) = Cx^\lambda + Dx^\kappa$. The conditions $w(y_1) = g(y_1)$ and $\lim_{x \to \infty} w(x) = 0$ imply $D = 0$ and $C = g(y_1) (1/y_1)^\lambda$, so $w(x) = g(y_1) (x/y_1)^\lambda$. Next, note that for all $x_0 > y_1$, $\tau_y \to \tau_{y_1}$ as $y_2 \to \infty$. By Dominated convergence, $W(x) = E \left[ e^{-r\tau_y} g(x_{\tau_y}) \mid x_0 = x \right] \to E \left[ e^{-r\tau_{y_1}} g(x_{\tau_{y_1}}) \mid x_0 = x \right]$ as $y_2 \to \infty$. By Lemma B1, $W(x) = Ax^\lambda + Bx^\kappa$ for $x \in (y_1, y_2)$, with $A$ and $B$ satisfying (B.2). Since $\lim_{y_2 \to \infty} B = 0$ and $\lim_{y_2 \to \infty} A = g(y_1)/y_1^\lambda$, $E \left[ e^{-r\tau_{y_1}} g(x_{\tau_{y_1}}) \mid x_0 = x \right] = \lim_{y_2 \to \infty} W(x) = w(x)$ for all $x > y_1$. $lacksquare$

Proof of Lemma 1. Let $V_k(\cdot)$ be as in the statement of the Lemma. Note that $V_k$ is twice differentiable with a continuous first derivative. One can show that $V_k(x) > v_k - x$ for $x > z_k$, so $V_k(x) \geq v_k - x$ for all $x \geq 0$. Note also that $V_k(\cdot)$ solves (B.1) for all $x > z_k$, with $V_k(z_k) = v_k - z_k$ and $\lim_{x \to \infty} V_k(x) = 0$. By Corollary B1, $V_k(x) = E[e^{-r\tau_y} (v_k - x_{\tau_y}) \mid x_0 = x]$. Moreover, $r(v_k - x) = rV_k(x) + \mu x V'_k(x) + \frac{1}{2} \sigma^2 x^2 V''_k(x) = -\mu x$, for all $x \leq z_k$. Therefore, $V_k$ is twice differentiable with a continuous first derivative, and satisfies

$$-rV_k(x) + \mu x V'_k(x) + \frac{1}{2} \sigma^2 x^2 V''_k(x) \leq 0,$$

with equality on $(z_k, \infty)$. Then, by standard verification theorems (Theorem 3.17 in Shiryaev, 2008) $V_k$ solves (6). $lacksquare$
Remark B1 Since $V_k$ is a solution to the optimal stopping problem \((6)\), then $e^{-rt}V_k(x_t)$ is superharmonic; i.e., $V_k(x) \geq E[e^{-rt}V_k(x_{\tau})|x_0 = x]$ for any stopping time $\tau$ (e.g., Theorem 10.1.9 in Oksendal, 2008). I will use this property of $V_k$ in the proof of Lemma 3 below.

Proof of Lemma 2. Equation (9) follows from Corollary B1. Moreover, for all $x \in (z_1, z_2]$, 

$$P(x, \alpha) - x - V_1(x) = v_2 - x - E[e^{-r\tau}v_2 - v_1|\tau = x] - E[e^{-r\tau}(v_1 - x_1)|\tau = x]$$

$$= v_2 - x - E[e^{-r\tau}(v_2 - x_2)|\tau = x] > 0$$

since by Lemma 1, $v_2 - x = V_2(x) > E[e^{-r\tau}(v_2 - x_2)|\tau = x]$ for all $x \in (z_1, z_2]$. □

B.2 Proof of Lemma 3

The proof of Lemma 3 is organized as follows. Lemmas B2 and B3 give properties of solutions to equation (B.1). Lemma B4 uses these properties to characterize the solution to the optimal stopping problem (10). Finally, Lemmas B5 and B6 prove additional properties of (10).

Lemma B2 Let $U$ and $\tilde{U}$ be two solutions to (B.1). If $\tilde{U}(y) \geq U(y)$ and $\tilde{U}'(y) > U'(y)$ for some $y > 0$, then $\tilde{U}'(x) > U'(x)$ for all $x > y$, and so $\tilde{U}(x) > U(x)$ for all $x > y$. Similarly, if $\tilde{U}(y) \leq U(y)$ and $\tilde{U}'(y) > U'(y)$ for some $y > 0$, then $\tilde{U}'(x) > U'(x)$ for all $x < y$, and so $\tilde{U}(x) < U(x)$ for all $x < y$.

Proof. I prove the first statement of the Lemma. The proof of the second statement is symmetric and omitted. Suppose the claim is not true, and let $y_1 > y$ be the smallest point with $U'(y_1) = \tilde{U}'(y_1)$. Therefore, $\tilde{U}'(x) > U'(x)$ for all $x \in [y, y_1)$, so $\tilde{U}(y_1) > U(y_1)$. Since $U$ and $\tilde{U}$ solve (B.1), then $\tilde{U}''(y_1) = 2(r\tilde{U}(y_1) - \mu y_1\tilde{U}'(y_1))/\sigma^2 y_1^2 > 2(rU(y_1) - \mu y_1U'(y_1))/\sigma^2 y_1^2 = U''(y_1)$. But this implies that $U'(y_1 - \varepsilon) > \tilde{U}'(y_1 - \varepsilon)$ for $\varepsilon > 0$, a contradiction. □

Lemma B3 Fix $q \in [0, \alpha)$ and $y \in (0, z_1)$, and let $U_q(y)$ be the solution to (B.1) with $U_q(y) = (1 - q)(v_1 - y)$. Then, $U_q(x)$ is strictly convex for all $x > 0$. Moreover, if $y < y' < z_1$, then $U_q(y) < U_q(y')$ for all $x \geq y'$.

Proof. Since $U_q(\cdot)$ solves (B.1), it follows that $U_q(y) = Ax^\lambda + Bx^\kappa$. The constants $A$ and $B$ are determined by the conditions $U_q(y) = (1 - q)(v_1 - y)$ and $U_q'(y) = -(1 - q)$: $A = y^\lambda \frac{(1 - q)(v_1 - y)}{\kappa + \lambda} > 0$ and $B = y^\kappa \frac{(1 - q)(v_1 - y)^\lambda - y}{\kappa + \lambda} > 0$, where the second inequality follows from the fact that $y < z_1 = v_1\lambda/(1 - \lambda)$. Thus, $U_q''(x) = \lambda(\lambda - 1)Ax^{\lambda - 2} + \kappa(\kappa - 1)Bx^{\kappa - 2} > 0$ for all $x > 0$ (since $\kappa > 1$). Finally, let $y < y' < z_1$. Since $U_q(\cdot)$ is strictly convex, it follows that $U_q(y') = (1 - q)(v_1 - y') = U_q(y')$ and $U_q'(y') = -(1 - q) = U_q'(y')$. Hence, by Lemma B2 $U_q(x) < U_q(y')$ for all $x \geq y'$. □

For $q \in [0, \alpha)$ and $x > 0$, let $g(x, q) = (\alpha - q)(P(x, \alpha) - x) + \Pi(x, \alpha)$. Thus, $L(x, q) = \inf \{x \in [0, \bar{x}(q)] \cup [\bar{x}(q), z_1] \}$. Note that $g(x, q) = (1 - q)(v_1 - y)$ for all $x \leq z_1$.

Lemma B4 For all $q \in [0, \alpha)$, there exists $\bar{x}(q) \in (0, z_1)$ and $\bar{x}(q) \in (z_1, z_2)$ such that $\tau(q) = \inf \{t : x_t \in [0, \bar{x}(q)] \cup [\bar{x}(q), z_2] \}$ solves (10). Moreover,
By Lemma B.3, the point \( y \) is close to \( (x,q) \), for all \( x \in \{ x : q \} \cup \{ 0, \infty \} \), \( L(x,q) \) solves (B.1), with \( \lim_{x \to \infty} L(x,q) = 0 \).

(ii) for all \( x \leq \bar{x}(q) \) and all \( x \in \{ \bar{x}(q) : q \} \), \( L(x,q) = g(x,q) \).

(iii) the cutoffs \( \bar{x}(q) \) and \( \bar{\pi}(q) \) are such that

\[
L(\bar{x}(q),q) = g(\bar{x}(q),q), \quad L(\bar{\pi}(q),q) = g(\bar{\pi}(q),q),
\]

(VM)

\[
L_x(\bar{x}(q),q) = g_x(\bar{x}(q),q), \quad L_x(\bar{\pi}(q),q) = g_x(\bar{\pi}(q),q).
\]

(SP)

**Proof.** First I show that there exists a function \( G(x,q) \) satisfying (i)-(iii). Then I show that \( G(x,q) = \sup_x E[\exp(x,q)|x_0 = x] = L(x,q) \). Let \( W(x) \) be the solution to (B.1) with \( \lim_{x \to \infty} W(x) = 0 \) and \( W(z_2) = g(z_2,q) \). By Corollary B.1, \( W(x) = g(z_2,q) (x/z_2)^\lambda \). For future reference, note that \( W'(z_2) = g_z(z_2,q) \). Moreover, one can check that \( W(x) > g(x,q) \) for all \( x > z_2 \) and that \( g(x,q) < W(x) \) for all \( x < z_1 \). For all \( x \geq z_2 \), let \( G(x,q) = W(x) \).

Next, I show that there exists a function \( G(x,q) \) and unique cutoffs \( \bar{x}(q) < z_1 \) and \( \bar{\pi}(q) \) in \( (z_1,z_2) \) such that \( G(x,q) \) solves (B.1) on \( (\bar{x}(q),\bar{\pi}(q)) \) and satisfies (iii). To see this, consider solutions \( U \) to (B.1) with \( U(y) = g(y,q) = (1 - q) (v_1 - y) \) and \( U'(y) = g_x(y,q) = -(1 - q) \) for some \( y < z_1 \). By Lemma B.3, such solutions are strictly convex. Since solutions to (B.1) are continuous in initial conditions, then the solutions I’m considering are continuous in \( y \). If \( y \) is small enough, then \( U(x) \) will remain above \( g(x,q) \) for all \( x > y \). On the other hand, if \( y \) is close to \( z_1 \) then \( U \) will cross \( g(x,q) \) at some \( \bar{x} > z_1 \) (see solutions I-IV in Figure B1). By Lemma B.3, the point \( \bar{x} \) moves to the right as \( y \) decreases. Let \( \bar{x}(q) \) be the smallest \( y \) such that \( U \) reaches \( g(x,q) \) at some \( \bar{\pi}(q) > y \). Since a solution with \( y < \bar{x}(q) \) never reaches \( g(x,q) \), it follows that \( U(x) \geq g(x,q) \) for all \( x \). Thus, \( U \) is tangent to \( g(x,q) \) at \( \bar{\pi}(q) \), so \( U'(\bar{\pi}(q)) = g_x(\bar{\pi}(q),q) \) (solution III in Figure B1). Let \( G(x,q) = U(x) \) for \( x \in [\bar{x}(q),\bar{\pi}(q)] \).

By construction, it must be that \( \bar{x}(q) \) is in \( (0,z_1) \) and that \( \bar{\pi}(q) > z_1 \). Moreover, it must be that \( \bar{\pi}(q) < z_2 \). To see this, suppose by contradiction that \( \bar{\pi}(q) \geq z_2 \). If \( \bar{\pi}(q) = z_2 \), then \( U(x) \) and \( W(x) \) both solve (B.1), with \( U(z_2) = W(z_2) = U'(z_2) = W'(z_2) \). Hence, \( W(x) = U(x) \) for all \( x \), which cannot be since \( U(x) = Ax^\lambda + Bx^\kappa \) for some \( B > 0 \) and \( W(x) = g(z_2,q) (x/z_2)^\lambda \). On the other hand, if \( \bar{\pi}(q) > z_2 \), then \( W(\bar{\pi}(q)) > U(\bar{\pi}(q)) = g(\bar{\pi}(q),q) \) and \( W(z_2) = g(z_2,q) < U(z_2) \). Moreover, since \( W(x) > g(x,q) \) for all \( x < z_1 \), it follows that \( U(\bar{x}(q)) < W(\bar{x}(q)) \). By the intermediate value theorem, there exists \( y_1 \in (\bar{x}(q),z_2) \) and \( y_2 \in (z_2,\bar{\pi}(q)) \) with \( U(y_i) = W(y_i) \) for \( i = 1,2 \). This implies that \( U(x) = W(x) \) for all \( x \), since \( U \) and \( W \) solve (B.1) on \( (y_1,y_2) \) with \( U(y_i) = W(y_i) \) for \( i = 1,2 \). But again this cannot be, so \( \bar{\pi}(q) < z_2 \).

For all \( x \leq \bar{x}(q) \) and for all \( x \in [\bar{x}(q),z_2] \) let \( G(x,q) = g(x,q) \). Note that, by construction, \( G(x,q) \) satisfies (i)-(iii).

I now show that \( G(x,q) = L(x,q) = \sup_x E[\exp(x,q)|x_0 = x] \). By construction, \( G(x,q) \geq g(x,q) \) for all \( x \geq 0 \). Moreover, \( G(x,q) \) is twice differentiable in \( x \), with a continuous first derivative. Finally, the function \( G(x,q) \) satisfies:

\[
-r G(x,q) + \mu x G_x(x,q) + \frac{1}{2} \sigma^2 x^2 G_{xx}(x,q) \leq 0, \quad \text{with equality on } (\bar{x}(q),\bar{\pi}(q)) \cup (z_2,\infty).
\]
Indeed, $G(x, q)$ satisfies (B.3) with equality on $(\underline{x}(q), \overline{x}(q)) \cup (z_2, \infty)$ since it solves (B.1) in this region. One can also check that $rG(x, q) \geq \mu x G_x(x, q) + \frac{1}{2} \sigma^2 x^2 G_{xx}(x, q)$ for all $x \in [0, \underline{x}(q)] \cup [\overline{x}(q), z_2]$. Therefore, by standard verification theorems (e.g., Theorem 3.17 in Shiryaev, 2008), $G(x, q) = \sup_{\tau} E[e^{-\tau r} g(x, q)|x_0 = x] = L(x, q)$. By Lemma B1 and Corollary B1, $L(x, q) = E[e^{-\tau r} g(x(q), q)|x_0 = x]$, so $\tau(q)$ solves (10). ■

**Lemma B5** $L(x, q) \in C^2$ for all $x \in (\underline{x}(q), \overline{x}(q))$ and all $q \in [0, \alpha)$. Moreover, $\underline{x}(q)$ and $\overline{x}(q)$ are $C^2$, with $\lim_{q \to \alpha} \underline{x}(q) = \lim_{x \to \infty} \overline{x}(q) = z_1$.

**Proof.** By Lemma B4, $L(x, q) = A(q) x^\lambda + B(q) x^\kappa$ for all $x \in (\underline{x}(q), \overline{x}(q))$, where $A(q), B(q), \underline{x}(q)$ and $\overline{x}(q)$ are determined by the system of equations (VM) + (SP). Denote this system of equations by $F(\underline{x}(q), \overline{x}(q), A(q), B(q)) = 0$. One can check that $F \in C^2$ and its Jacobian at $(\underline{x}(q), \overline{x}(q), A(q), B(q))$ has a non-zero determinant. By the Implicit Function Theorem, the functions $A(q), B(q), \underline{x}(q)$ and $\overline{x}(q)$ are all $C^2$ with respect to $q$ (e.g., de la Fuente, 2000, pages 210-211). Since $L(x, q) = A(q) x^\lambda + B(q) x^\kappa$ for all $x \in (\underline{x}(q), \overline{x}(q))$, it follows that $L(x, q) \in C^2$ for all $x \in (\underline{x}(q), \overline{x}(q))$.\[1\]

Next, I show that $\lim_{q \to \alpha} \underline{x}(q) = \lim_{q \to \alpha} \overline{x}(q) = z_1$. Let $\underline{x} = \lim_{q \to \alpha} \underline{x}(q)$ and $\overline{x} = \lim_{q \to \alpha} \overline{x}(q)$. Since $x(q) < z_1$ and $\overline{x}(q) > z_1$ for all $q$ (Lemma B4), it follows that $\underline{x} \leq z_1 \leq \overline{x}$. Let $\overline{\tau} = \inf\{t : x_t \in [0, \underline{x}] \cup [\overline{x}, z_2]\}$, so $\tau(q_n) \to \overline{\tau}$ for every sequence $\{q_n\} \to \alpha$. Note that $L(x, q) \geq g(x, q) \geq \Pi(x, \alpha) = (1 - \alpha) V_1(x)$ for all $q \leq \alpha$, so $\lim_{q \to \alpha} L(x, q) \geq (1 - \alpha) V_1(x)$. Let $\{q_n\} \to \alpha$. Since $\lim_{q \to \alpha} g(x, q) = (1 - \alpha) V_1(x)$, by Dominated Convergence, $L(x, q_n) = E[e^{-\tau r(q_n)} g(x(q_n), q_n)|x_0 = x] \to E[e^{-\overline{\tau} r} (1 - \alpha) V_1(x)|x_0 = x]$ as $n \to \infty$. Suppose by contradiction that $\underline{x} < z_1$. Then, for $x \in (\underline{x}, \overline{x})$,

$$E[e^{-\overline{\tau} r} V_1(x)|x_0 = x] = \Pr(x_{\overline{\tau}} = \underline{x}) E[e^{-\tau r}(v_1 - \underline{x})|x_0 = x] + \Pr(x_{\overline{\tau}} = \overline{x}) E[e^{-\tau r}(\overline{x}) V_1(x)|x_0 = x] < V_1(x),$$

\[1\] Thus, $L_q(x, q) = A'(q) x^\lambda + B'(q) x^\kappa$ for all $x \in (\underline{x}(q), \overline{x}(q))$, so $L_q(x, q)$ solves (B.1) in this range.
where the inequality follows from Remark B1 and the fact that \( E[e^{-rt}(x_1 - x)] < V_1(x) = E[e^{-rt}(v_1 - z)] \) (by Lemma 1). Hence, \( (1 - \alpha)E[e^{-rt}V_1(x)|x_0 = x] < (1 - \alpha) V_1(x) \), which contradicts \( \lim_{n \to \alpha} L(x,q) \geq (1 - \alpha) V_1(x) \). Thus, it must be that \( z = z_1 \).

Suppose next that \( \bar{z} > z_1 \). Let \( W(x) = E[e^{-rt}(P(x,\alpha) - x)]|x_0 = x] \) and \( Y_t = e^{-rt}(P(x_t,\alpha) - x_t) \). By Ito’s Lemma, for all \( x \in (z_1,\bar{z}) \),

\[
dY_t = e^{-rt}\left( -r(P(x_t,\alpha) - x_t) + \mu x_t(P_x(x_t,\alpha) - 1) + \frac{\sigma^2 x^2}{2}P_{xx}(x_t,\alpha) dt + \sigma x_t P_x(x_t,\alpha) dB_t \right)
\]

where the second equality follows since equation (9) implies that \( rP(x,\alpha) = rv_2 + \mu x P_x(x,\alpha) + \frac{\sigma^2 x^2}{2}P_{xx}(x,\alpha) \) for all \( x > z_1 \). Therefore, for \( x \in (z_1,\bar{z}) \),

\[
W(x) = E[Y_\bar{z}|x_0 = x] = Y_0 + E\left[ \int_0^\bar{z} e^{-rt}(-r(v_2 - x_t) - \mu x_t) dt \right]|x_0 = x.
\]

One can check that \(-r(v_2 - x) < \mu x \) for all \( x < \bar{z} < z_2 \), so \( W(x) < Y_0 = P(x,\alpha) - x \).

For each \( q \in [0,\alpha) \), let \( W(x,q) = E[e^{-r\tau(q)}(P(x,\alpha) - x(q))|x_0 = x] \). Pick a sequence \( \{q_n\} \to \alpha \), so that \( \tau(q_n) \to \bar{\tau} \) as \( n \to \infty \). By dominated Convergence, \( W(x,q_n) \to W(x) \) as \( n \to \infty \). Fix \( x \in (z_1,\bar{z}) \). Since \( W(x) < P(x,\alpha) - x \), there exists \( N \) such that \( W(x,q_n) < P(x,\alpha) - x \) for all \( n > N \). On the other hand, \( E[e^{-r\tau(q_n)}V_1(x(q_n))|x_0 = x] \leq V_1(x) \) for all \( x \) and all \( n \) (see Remark B1). Therefore, for \( n > N \)

\[
L(x,q) = E\left[ e^{-r\tau(q_n)} \left( (\alpha - q_n)(P(x(q_n),\alpha) - x(q_n)) + (1 - \alpha)V_1(x(q_n)) \right)|x_0 = x \right] < (\alpha - q_n)(P(x,\alpha) - x) + (1 - \alpha)V_1(x) = g(x,q_n),
\]

which contradicts the fact that \( L(x,q_n) = \sup_{\tau} E[e^{-r\tau}g(x,\tau,q_n)|x_0 = x] \). Thus, \( \bar{z} = z_1 \). □

**Proof of Lemma 3.** Follows immediately from Lemmas B4 and B5. □

**Lemma B6** \( L(x,q) \) is strictly convex in \( q \) for all \( x \in (\bar{x}(q),\bar{x}(q)) \). Moreover, \( \bar{x}'(q) > 0 \) and \( \bar{x}''(q) < 0 \).

**Proof.** I first show that \( \bar{x}'(q) > 0 \) and \( \bar{x}''(q) < 0 \). For \( q \in [0,\alpha) \), let \( W(x,q) = E[e^{-r\tau(q)}(P(x,\alpha) - x(q))|x_0 = x] \) and \( U(x,q) = E[e^{-r\tau(q)}V_1(x(q))|x_0 = x] \), so \( L(x,q) = (\alpha - q)W(x,q) + (1 - \alpha)U(x,q) \). By Lemma B1, \( U(x,q) \) solves (B.1) for all \( x \in (\bar{x}(q),\bar{x}(q)) \) with \( U(\bar{x}(q),q) = v_1 - \bar{x}(q) = V_1(\bar{x}(q)) \) and \( U(\bar{x}(q),q) = (v_1 - z_1)(\bar{x}(q) - z_1)^{\lambda} = V_1(\bar{x}(q)) \). I now show that \( U_x(\bar{x}(q),q) < U'_x(\bar{x}(q)) = -1 \) and \( U_x(\bar{x}(q),q) > V'_1(\bar{x}(q)) \). To see this, note that \( V_1(x) \) also solves (B.1) for \( x \geq z_1 \), with \( V_1(z_1) = v_1 - z_1 \) and \( V'_1(z_1) = -1 \). Suppose by contradiction that \( U_x(\bar{x}(q),q) \geq -1 \). By Lemmas B2 and B3, \( U''(x) > -1 \) and \( U(x) \geq v_1 - x \) for all \( x > \bar{x}(q) \). Lemma B2 then implies that \( U(x,q) > V_1(x) \) for all \( x > \bar{x}(q) \), a contradiction to the fact that \( U(\bar{x}(q),q) = V_1(\bar{x}(q)) \). Hence, \( U_x(\bar{x}(q),q) < -1 \). A symmetric argument establishes that \( U_x(\bar{x}(q),q) > V'_1(\bar{x}(q)) \). Since \( L_x(\bar{x}(q),q) = g_x(\bar{x}(q),q) = \)
Let $q < q' < \alpha$. Fix $q < q'$. Let $F(x) = g(q(x), q') = (1 - q') (v_1 - x(q))$ and $F'(x) = g_x(q(x), q') = - (1 - q')$. By Lemma B2, $F(x) \geq H(x) \geq g(x, q') = (1 - q')$ for all $x \in [x(q), z_1]$. Since $F$ solves (B.1) with $F(q(x)) = g(q(x), q')$ and $F'(x) > g_x(q(x), q')$, it follows by Lemma B2 that $F(x) \geq H(x) \geq g(x, q')$ and $F'(x) > H'(x) \geq g_x(x, q')$ for all $x \in [x(q), z_1]$. By Lemma B4, $L(x, q')$ solves (B.1) on $(q(x), F(q(x)))$, with $L(x, q', q') = g(q(x), q') \leq F(q(x))$ and $L_x(q(x), q') = g_x(q(x), q') < F'(q(x))$. Lemma B2 then implies that $L(x, q') < F(x) = E[e^{-\gamma q} g(x_{\tau(q)}, q')] | x_0 = x]$. Thus, $\pi(q') < \pi(q)$.

Similarly, suppose that $\pi(q') \leq \pi(q)$. By a symmetric argument, one can show that $L(\pi(q'), q') = g(\pi(q'), q') \leq F(\pi(q'))$ and $L_x(\pi(q'), q') = g_x(\pi(q'), q') > F_x(\pi(q'))$. Lemma B2 then implies that $F(x) > L(x, q')$ for all $x < \pi(q')$, contradicting the fact that $L(x, q') = \sup \pi E[e^{-\gamma q} g(x_{\tau(q)}, q')] | x_0 = x]$. Thus, $\pi(q') > \pi(q)$.

Finally, I show that $L(x, q)$ is strictly convex in $q$ for all $x \in [x(q), \pi(q)]$. Take $q' < q < \alpha$, and let $q_t = \gamma q + (1 - \gamma) q'$ for some $\gamma \in (0, 1)$. Note that $g(x, q_t) = \gamma g(x, q) + (1 - \gamma) g(x, q')$. Moreover, $\pi(q_t) < \pi(q) < \pi(q')$. Therefore,

$$L(x, q) = \gamma E[e^{-\gamma t} g(x_{\tau(q_t)}, q') | x_0 = x] + (1 - \gamma) E[e^{-\gamma t} g(x_{\tau(q_t)}, q') | x_0 = x] < \gamma L(x, q) + (1 - \gamma) L(x, q')$$

for all $x \in (x(q), \pi(q))$, so $L(x, q)$ is strictly convex in $q$ on $(x(q), \pi(q))$.

**B.3 Supplement to proof of Theorem 1**

In this Appendix I show that in any equilibrium the monopolist’s profits at state $(x, q)$ with $q < \alpha$ are equal to $L(x, q)$. The proof is divided in a series of Lemmas.

**Lemma B7** Let $\{q_t\}, P$ be an equilibrium. Then,

(i) for all $t$ such that $x_t \leq z_2$ and $q_t < \alpha$, the monopolist always sells (i.e., $d_{q_t} > 0$),

(ii) for all $t$ such that $x_t > z_2$ and $q_t < \alpha$, the monopolist doesn’t sell (i.e., $d_{q_t} = 0$).

**Proof.** (i) Suppose that the monopolist doesn’t sell while $x_t \leq z_2$. Let $\tau = \inf \{s > t : q_s > q_t\}$, so $\tau > t$. By payoff maximization, the price that the marginal buyer $q_t^+ = \lim_{q_t \uparrow} q_t + \epsilon$ is willing to pay at time $t$ satisfies $P(x_t, q_t^+) = v_2 - E_t[e^{-\tau(t-\tau)}(v_2 - P(x_{\tau}, q_{\tau}))]$. The monopolist gets a profit margin of $E_t[e^{-\tau(t-\tau)}(P(x_{\tau}, q_{\tau}) - x_{\tau})]$ from selling to $q_t^+$ at time $\tau$, while she
would get $P(x_t, q_t^+)$ at time $t$. Note that

$$P(x_t, q_t^+ - x_t - E_t[e^{-r(\tau - t)}(P(x, q) - x)] = v_2 - x_t - E_t[e^{-r(\tau - t)}(v_2 - x_t)] > 0,$$

where the inequality follows since $v_2 - x_t > E_t[e^{-r(\tau - t)}(v_2 - x_t)]$ for all $\tau > t$ when $x_t \leq z_2$ (Lemma 1). Thus, the seller is better off selling to $q_t^+$ at $t$. Finally, since $P(x, i)$ has a right-hand limit, there exists $\varepsilon > 0$ such that the monopolist is better off selling to all $i \in (q_t^+, q_t^+ + \varepsilon]$ at time $t$ than after time $\tau$, so $(\{q_t\}, P)$ cannot be an equilibrium.

(ii) Suppose that the monopolist sells while $x_t > z_2$. Let $\tau_n$ denote the time at which consumer $\alpha$ buys and recall that $\tau_2 = \inf\{t : x_t \leq z_2\}$. Let $\tau = \min\{\tau_0, \tau_2\}$. I first show that the price at which the monopolist sells at any $s \in [t, \tau_2]$ satisfies $P(x_s, q_s) = v_2 - E_s[e^{-r(t_2-s)}(v_2 - P(x_{\tau_2}, q_{\tau_2}))]$. To see this, note that all high types must get the same payoff in equilibrium. Hence, for all $u \in [s, \tau_2]$, the price at which the monopolist sells at $s$ satisfies $P(x_s, q_s) = v_2 - E_s[e^{-r(u-s)}(v_2 - P(x_u, q_u))]$. Thus, if $\tau_0 \geq \tau_2$, then $P(x_s, q_s) = v_2 - E_s[e^{-r(t_2-s)}(v_2 - P(x_{\tau_2}, q_{\tau_2}))]$ for all $s \in [t, \tau_2]$. On the other hand, if $\tau_0 < \tau_2$, then $P(x_s, q_s) = v_2 - E_s[e^{-r(\tau_0-s)}(v_2 - P(x_{\tau_0}, q_{\tau_0}))]$ for all $s \in [t, \tau_0]$. By equation (8),

$$P(x_{\tau_0}, q_{\tau_0}) = v_2 - E_{\tau_0} \left[e^{-r(\tau_1-\tau_0)}(v_2 - v_1)\right] = v_2 - E_{\tau_0} \left[e^{-r(\tau_2-\tau_0)}(v_2 - P(x_{\tau_2}, \alpha))\right],$$

where the second equality follows since $P(x_{\tau_2}, \alpha) = v_2 - E \left[e^{-r(t_1-t_2)}(v_2 - v_1)\right]$. The law of iterated expectations and the fact that $q_{\tau_2} = \alpha$ whenever $\tau_0 < \tau_2$ imply that $P(x_s, q_s) = v_2 - E_s[e^{-r(\tau_0-s)}(v_2 - P(x_{\tau_0}, \alpha))] = v_2 - E_s[e^{-r(t_2-s)}(v_2 - P(x_{\tau_2}, q_{\tau_2}))]$.

The profits that the monopolist gets from selling to high valuation consumers between time $t$ and $\tau_2$ are $E_t[e^{-r(s-t)}\int_t^{\tau_2}(P(x_s, q_s) - x_s)ds]$. If instead the monopolist waits until time $\tau_2$ and sells to all consumers $i \in [q_{t-}, q_{\tau_2}]$ at that instant, her profits are $E_t[e^{-r(t_2-t)}(P(x_{\tau_2}, q_{\tau_2}) - x_{\tau_2})(q_{\tau_2} - q_{t-})]$. Since $P(x_s, q_s) = v_2 - E_s[e^{-r(t_2-s)}(v_2 - P(x_{\tau_2}, q_{\tau_2}))]$ for all $s \in [t, \tau_2],

$$P(x_s, q_s) - x_s - E_s \left[e^{-r(t_2-s)}(P(x_{\tau_2}, q_{\tau_2}) - x_{\tau_2})\right] = v_2 - x_s - E_s \left[e^{-r(t_2-s)}(v_2 - x_{\tau_2})\right] < 0,$$

since, by Lemma 1, $v_2 - x_s < E_s[e^{-r(t_2-s)}(v_2 - x_{\tau_2})]$ for all $x_s > z_2$. Hence, the seller is better off by delaying sales until $\tau_2$, so $(\{q_t\}, P)$ cannot be an equilibrium.

**Lemma B8.** Let $(\{q_s\}, P)$ be an equilibrium and let $\Pi(x, q)$ be the seller’s profits. If $\{q_s\}$ is continuous in $[t, \tau)$ for some $\tau > t$, then $\Pi(x_t, q_t) = E_t[e^{-r(u-t)}\Pi(x_u, q_u)]$ for all $u \in [t, \tau]$.

**Proof.** Suppose first that $\{q_s\}$ is constant on $[t, \tau)$. Then, $\Pi(x_t, q_t) = E_t[e^{-r(u-t)}\Pi(x_u, q_u)]$ for all $u \in [t, \tau]$, since the monopolist doesn’t make any sales on $[t, \tau)$.

Suppose next that $\{q_s\}$ is continuous and strictly increasing in $[t, \tau)$. Fix $u \in [t, \tau]$. Since the monopolist can always choose to make no sales between $t$ and $u > t$, it follows that $\Pi(x_t, q_t) \geq E_t[e^{-r(u-t)}\Pi(x_u, q_u)]$. Suppose by contradiction that $\Pi(x_t, q_t) >}

---

\footnote{This follows since, by the skimming property, $P(x, i)$ is monotonically decreasing in $i$.}
\[ E_t[e^{-r(u-t)}\Pi(x_u,q_t)] \] for some \( u \in [t, \tau] \). Therefore, it must be that

\[ \Pi(x_t,q_t) = E_t \left[ \int_t^u e^{-r(s-t)} (P(x_s,q_s) - x_s) dq_s + e^{-r(u-t)}\Pi(x_u,q_u) \right] > E_t \left[ e^{-r(u-t)}\Pi(x_u,q_t) \right]. \]

Note that \( \Pi(x_u,q_t) \geq \int_q^u (P(x,u,i) - x_u) di + \Pi(x_u,q_u) \), since at state \((x_u,q_t)\) the monopolist can get profits arbitrarily close to \( \int_q^u (P(x,u,i) - x_u) di + \Pi(x_u,q_u) \) by selling to all buyers \( i \in [q_t, q_u] \) arbitrarily fast. It then follows that

\[ E_t \left[ \int_t^u e^{-r(s-t)} (P(x_s,q_s) - x_s) dq_s \right] > E_t \left[ e^{-r(u-t)} \int_q^u (P(x,u,i) - x_u) di \right]. \quad (B.4) \]

Equation (B.4) in turn implies that there exists a set of positive measure \([q, \bar{q}] \subset [q_t, q_u] \) and \( s_1, s_2 \in [t, u] \) with \( s_2 > s_1 \) such that \( P(x_s,i) - x_s > E_s[e^{-r(s'-s)} (P(x',i) - x')] \) for all \( i \in [q, \bar{q}] \) and all \( s', s \in [s_1, s_2] \). Pick \( \varepsilon > 0 \) small enough such that \( \tau_{\varepsilon} = \inf \{ s : q_s \geq q_0 + \varepsilon \} \leq s_2 \) whenever the state at time \( s_1 \) is \((x_{s_1},q)\). Then, it follows that

\[ \Pi(x_{s_1},q) = E_{s_1} \left[ \int_{s_1}^{\tau_{\varepsilon}} e^{-r(s-s_1)} (P(x,s,q_s) - x_s) dq_s + e^{-r(\tau_{\varepsilon}-s_1)}\Pi(x_{\tau_{\varepsilon}},q_{\tau_{\varepsilon}}) \right] \]

\[ < \int_q^{q+\varepsilon} (P(x_{s_1},i) - x_{s_1}) di + E_{s_1} \left[ e^{-r(\tau_{\varepsilon}-s_1)}\Pi(x_{\tau_{\varepsilon}},q_{\tau_{\varepsilon}}) \right]. \quad (B.5) \]

At state \((x_{s_1},q)\) the monopolist can get a payoff arbitrarily close to the right-hand side of (B.5) by selling to all \( i \in [q, q + \varepsilon] \) arbitrarily fast and then not making any sales until time \( \tau_{\varepsilon} \). Thus, the seller has a strategy that yields her a higher payoff than \((q_t, P)\), which cannot be since \((\{q_t\}, P)\) is an equilibrium. Hence, \( \Pi(x_t,q_t) = E_t[e^{-r(s-t)}\Pi(x_s,q_t)] \) for all \( s \in [t, \tau] \).

**Lemma 9** Let \((\{q_t\}, P)\) be an equilibrium and let \( \Pi(x,q) \) be the seller’s profits. Let \((x,q)\) with \( q < \alpha \) be such that \( \{q_t\} \) is continuous at time \( s \) when \((x_s,q_s) = (x,q)\). Then, there exists \( \tau > s \) such that \( \{q_t\} \) jumps at state \((x_{\tau},q)\). Moreover,

\[ \Pi(x_{\tau},q) = E_{\tau} \left[ e^{-r(\tau-s)} ((P(x_{\tau},q + dq_{\tau}) - x_{\tau}) dq_{\tau} + \Pi(x_{\tau},q + dq_{\tau})) \right], \]

where \( dq_{\tau} \) denotes the jump of \( \{q_t\} \) at state \((x_{\tau},q)\).

**Proof.** Let \((x_{s},q_s) = (x,q)\) be as in the statement of the Lemma. By Lemma B8, there exists \( \tau > s \) such that \( \Pi(x_{\tau},q_{\tau}) = E_{s}e^{-r(s-s)}\Pi(x_{\tau},q_{\tau}) \). There are two possibilities: (i) \( \{q_t\} \) jumps at state \((x_{\tau},q_{\tau})\), or (ii) \( \{q_t\} \) is continuous at such state. In case (i), it must be that \( \Pi(x_{\tau},q_{\tau}) = (P(x_{\tau},q + dq_{\tau}) - x_{\tau}) dq_{\tau} + \Pi(x_{\tau},q + dq_{\tau}) \), where \( dq_{\tau} \) denotes the jump of \( \{q_t\} \) at state \((x_{\tau},q_{\tau})\). Thus, \( \Pi(x_{\tau},q_{\tau}) = E_{\tau}e^{-r(\tau-s)}((P(x_{\tau},q + dq_{\tau}) - x_{\tau}) dq_{\tau} + \Pi(x_{\tau},q + dq_{\tau})) \), and so the statement of the Lemma is true.

Consider next case (ii), so that \( \{q_t\} \) is continuous at state \((x_{\tau},q_{\tau})\). By Lemma B8 there exists \( \tau' > \tau \) such that \( \Pi(x_{\tau'},q_{\tau}) = E_{\tau'}e^{-r(\tau'-\tau)}\Pi(x_{\tau'},q_{\tau}) \), so by the law of iterated expectations \( \Pi_{\tau}(x_{\tau},q_{\tau}) = E_{\tau}e^{-r(\tau'-\tau)}\Pi(x_{\tau},q_{\tau}) \). Again, there are two possibilities: \( \{q_t\} \) jumps at
state \((x_\tau', q_s)\), or (ii) \(\{q_t\}\) is continuous at such state. By the argument above, the statement of the Lemma is true if the relevant case is (i). Otherwise, in case (ii) there exists \(\tau'' > \tau'\) such that \(\Pi(x_s, q_s) = E_s[e^{-r(\tau''-s)}\Pi(x_{\tau''}, q_s)]\). Continuing this way, there must exist \(\tilde{\tau}\) such that \(\{q_t\}\) jumps at state \((x_{\tilde{\tau}}, q_s)\); otherwise, \(\Pi(x_s, q_s) = \lim_{\tau \to \infty} E_s[e^{-r(\tau-s)}\Pi(x_\tau, q_s)] = 0\), which contradicts \(\Pi(x_s, q_s) \geq L(x_s, q_s) > 0\). 

**Lemma B10** Let \((\{q_t\}, P)\) be an equilibrium and let \(\Pi(x, q)\) be the seller’s profits. If \(\{q_s\}\) is continuous and strictly increasing in \([t, \tau)\) for some \(\tau > t\), then \(-\Pi_q(x_s, q_s) = P(x_s, q_s) - x_s\) for all \(s \in [t, \tau)\).

**Proof.** Note first that for all \(\varepsilon > 0\) and for all \(s \in [t, \tau)\), it must be that

\[
\Pi(x_s, q_s) \geq (P(x_s, q_s + \varepsilon) - x_s) \varepsilon + \Pi(x_s, q_s + \varepsilon),
\]

since the monopolist can always choose at time \(s\) to sell to all buyers \(i \in [q_s, q_s + \varepsilon]\) at price \(P(x_s, q_s + \varepsilon)\). Next, I show that for all \(s \in [t, \tau)\) it must also be that

\[
\Pi(x_s, q_s) \leq (P(x_s, q_s) - x_s) \varepsilon + \Pi(x_s, q_s + \varepsilon),
\]

for all \(\varepsilon > 0\) small enough. To see this, let \(\tau_\varepsilon = \inf\{u : q_u \geq q_s + \varepsilon\}\), so that \(\Pi(x_s, q_s) = E_s[\int_{\tau_\varepsilon}^\tau e^{-r(u-s)}(P(x_u, q_u) - x_u) du + e^{-r(\tau_\varepsilon-s)}\Pi(x_{\tau_\varepsilon}, q_s + \varepsilon)]\). Since \(P(x_s, q_s) \geq P(x_s, i)\) for all \(i > q_s\), and since \(\Pi(x_s, q_s + \varepsilon) \geq E_s[e^{-r(\tau_\varepsilon-s)}\Pi(x_{\tau_\varepsilon}, q_s + \varepsilon)]\),

\[
\Pi(x_s, q_s) - (P(x_s, q_s) - x_s) \varepsilon - \Pi(x_s, q_s + \varepsilon)
\]

\[
\leq E_s \left[ \int_{\tau_\varepsilon}^\tau e^{-r(u-s)}(P(x_u, q_u) - x_u) du \right] - \int_{q_s}^{q_s+\varepsilon} (P(x_s, i) - x_s) di,
\]

To establish (B.7) it suffices to show that the right-hand side of (B.8) is less than zero. Towards a contradiction, suppose that \(E_s[\int_{\tau_\varepsilon}^\tau e^{-r(u-s)}(P(x_u, q_u) - x_u) du] > \int_{q_s}^{q_s+\varepsilon} (P(x_s, i) - x_s) di\). Then, there exists \(s_1, s_2 \in [s, \tau_\varepsilon]\) with \(s_1 < s_2\) and a set \([q, \bar{q}] \subset [q_s, q_s + \varepsilon]\) such that \(P(x_{s_1}, i) - x_{s_1} < E_{s_1}[e^{-r(s'-s_1)}(P(x_{s'}, i) - x_{s'})]\) for all \(i \in [q, \bar{q}]\) and all \(s' \in (s_1, s_2]\). Suppose that the state at time \(s_1\) is \((x_{s_1}, \bar{q})\), and fix \(\tau' \in (s_1, s_2]\) such that \(q_{\tau'} \leq \bar{q}\). The previous inequality then implies that

\[
\Pi(x_{s_1}, \bar{q}) = E_{s_1} \left[ \int_{s_1}^{\tau'} e^{-r(s'-s_1)}(P(x, q_s) - x_s) dq_s + e^{-r(\tau'-s_1)}\Pi(x_{\tau'}, q_{\tau'}) \right]
\]

\[
< E_{s_1} \left[ e^{-r(\tau'-s_1)} \left( \int_{q_{\tau'}}^{q_{\tau'}} (P(x_{\tau'}, i) - x_{\tau'}) di + \Pi(x_{\tau'}, q_{\tau'}) \right) \right].
\]

Note that at state \((x_{s_1}, \bar{q})\) the seller can get a payoff arbitrarily close to the right-hand side of (B.9) by not making any sales until time \(\tau'\), and then selling to all buyers \(i \in [q, q_{\tau'}]\) arbitrarily fast at time \(\tau'\). Thus, at state \((x_{s_1}, \bar{q})\) the seller has a strategy different from \(\{q_t\}\) that gives her a larger payoff than \(\Pi(x_{s_1}, \bar{q})\), contradicting the assumption that \((\{q_t\}, P)\) is
an equilibrium. Hence, (B.7) must hold for $\varepsilon$ small enough.

Finally, by (B.6) and (B.7), for $\varepsilon > 0$ small, $P(x_s, q_s + \varepsilon) - x_s \leq \frac{-\Pi(x_s, q_s + \varepsilon) - \Pi(x_s, q_s)}{\varepsilon} \leq P(x_s, q_s) - x_s$. Since $P(x, q)$ must be continuous in $q$ at $(x_s, q_s)$ (otherwise, prices would jump down at time $s$, and those consumers who buy at $s^-$ would be strictly better off by waiting), it follows that $-\Pi_q (x_s, q_s) = P(x_s, q_s) - x_s$. \(\blacksquare\)

**Lemma B11** Let $\{q_i\}, P$ be an equilibrium, and let $\Pi(x, q)$ denote the monopolist’s profits. Then, $\Pi(x, q) = L(x, q)$ for all states $(x, q)$ with $q < \alpha$.

**Proof.** By the arguments in the main text, $\Pi(x, q) \geq L(x, q)$ for all states $(x, q)$ with $q < \alpha$. By Lemma B9, for all $(x, q)$ such that $\{q_i\}$ is continuously increasing at time $s$ when $(x_s, q_s) = (x, q)$, there exists $\tau > s$ and $dq_r > 0$ such that $\Pi(x, q) = E_s[e^{-r(\tau-s)}(P(x_r, q + dq_r) - x_r) dq_r + \Pi(x_r, q + dq_r))]$.

If $q + dq_r = \alpha$, then $\Pi(x, q) = E_s[e^{-r(\tau-s)}g(x_r, q)] \leq L(x, q)$, so $\Pi(x, q) = L(x, q)$. On the other hand, if $dq_r + q = \tilde{q} < \alpha - q$, then $\Pi(x, q) = (P(x_r, \tilde{q}) - x_r)(\tilde{q} - q) + \Pi(x_r, \tilde{q})$. By Lemma B7, $\{q_i\}$ must be continuous and strictly increasing after time $\tau$ (since, by Lemma B7, $x_r \leq z_2$).

Lemma B10 then implies that $P(x_r, \tilde{q}) - x_r = -\Pi_q(x_r, \tilde{q})$. By Lemma B9, $\Pi(x_r, \tilde{q}) = E_r[e^{-r(\tau'-\tau)}(\Pi(x_{\tau'}, \tilde{q} + dq_{\tau'}) - x_{\tau'} dq_{\tau'} + \Pi(x_{\tau'}, \tilde{q} + dq_{\tau'}))]$ for some $\tau' > \tau$, where $dq_{\tau'}$ denotes the jump of $\{q_i\}$ at state $(x_{\tau'}, \tilde{q})$. Therefore, $P(x_r, \tilde{q}) - x_r = -\Pi_q(x_r, \tilde{q}) = E_r[e^{-r(\tau'-\tau)}(P(x_{\tau'}, q + dq_{\tau'}) - x_{\tau'})]$. Since $\Pi(x_r, q) = (P(x_r, q) - x_r)(q - q) + \Pi(x_r, q)$ and since $\Pi(x_r, q) = E_r[e^{-r(\tau'-\tau)}(P(x_{\tau'}, \tilde{q} + dq_{\tau'}) - x_{\tau'} dq_{\tau'} + \Pi(x_{\tau'}, \tilde{q} + dq_{\tau'}))]$, it follows that

$$\Pi(x_r, q) = E_r\left[e^{-r(\tau'-\tau)}(P(x_{\tau'}, q_{\tau'}) - x_{\tau'}) (dq_{\tau'} + \tilde{q} - q) + \Pi(x_{\tau'}, q_{\tau'})\right],$$

(B.10)

where $q_{\tau'} = \tilde{q} + dq_{\tau'}$. There are two possibilities: (i) $q_{\tau'} = \alpha$, or (ii) $q_{\tau'} < \alpha$. In the first case, $P(x_{\tau'}, q_{\tau'}) - x_{\tau'} (dq_{\tau'} + \tilde{q} - q) + \Pi(x_{\tau'}, q_{\tau'}) = g(x_{\tau'}, q)$. Using (B.10), this implies that $\Pi(x_{\tau'}, q) = E_r[e^{-r(\tau'-\tau)}g(x_{\tau'}, q)]$; and since $\Pi(x, q) = E_s[e^{-r(\tau-s)}\Pi(x_r, q)]$, by the Law of Iterated Expectations, $\Pi(x, q) = E_s[e^{-r(\tau'-\tau)}g(x_r, q)] \leq L(x, q)$, so $\Pi(x, q) = L(x, q)$. In the second case, $q_{\tau'} < \alpha$. Since $x_{\tau'} \leq z_2$, by Lemma B7 the monopolist must continue selling gradually at state $(x_{\tau'}, q_{\tau'})$. Then, by Lemma B9, there exists $\tau''$ such that

$$\Pi(x_{\tau'}, q_{\tau'}) = E_r\left[e^{-r(\tau''-\tau)}(P(x_{\tau''}, q_{\tau''}) - x_{\tau''}) (dq_{\tau''} + q_{\tau''}) + \Pi(x_{\tau''}, q_{\tau''})\right],$$

(B.11)

where $q_{\tau''} = dq_{\tau'} + q_{\tau'}$. Moreover, the same arguments as above imply that $P(x_{\tau'}, q_{\tau'}) - x_{\tau'} = E_{\tau'}[e^{-r(\tau''-\tau)}(P(x_{\tau''}, q_{\tau''}) - x_{\tau''})]$. Using this and (B.11) in (B.10), it follows that $\Pi(x_r, q) = E_r\left[e^{-r(\tau'-\tau)}(P(x_r, dq_{\tau'} + q - x_r) dq_{\tau'} + \Pi(x_r, dq_{\tau'} + q))\right]$, for some stopping time $\tau'$ and some $dq_{\tau'} > 0$. Again, there are two possibilities: (i) $dq_{\tau'} + q = \alpha$, or (ii) $dq_{\tau'} + q < \alpha$. In case (i), $P(x_r, dq_{\tau'}) - x_r$ if $dq_{\tau'} + q < \alpha$. In case (ii), we can again repeat the same argument. Eventually, there will be a stopping time $\tau$ such that $\Pi(x, q) = E_s[e^{-r(\tau-s)}g(x_r, q)] \leq L(x, q)$, so $\Pi(x, q) = L(x, q)$. \(\blacksquare\)
B.4 Proofs of Propositions 3 and 4

Proof of Proposition 3. Let $\mu < 0$ and fix a sequence $\sigma_n \to 0$. For each $n$, let $\lambda_n$ be the negative root of $\frac{1}{2}\sigma_n^2\lambda (\lambda - 1) + \mu \lambda = r$. For $k = 1, 2$, let $z_n^k = \frac{-\lambda_n}{1-\lambda_n}v_k$. Let $P^n(x, \alpha)$ be buyer $\alpha$’s strategy when $\sigma = \sigma_n$. By Theorem 1, for each $n$ there exists $\varepsilon_n^0(0) < z_n^0$ and $\bar{\varepsilon}_n(0) \in (z_n^1, z_n^0)$ such that the monopolist sells at $t = 0$ to all high types at price $P^n(x, \alpha)$ when $x_0 \in [\bar{\varepsilon}_n(0), z_n^0]$, and sells at $t = 0$ to all buyers at price $v_1$ when $x_0 \leq \varepsilon_n^0(0)$. Let $\bar{x}^* = \lim_{n \to \infty} \varepsilon_n^0(0)$ and $\bar{\varepsilon} = \lim_{n \to \infty} \bar{\varepsilon}_n(0)$. Since $\lim_{n \to \infty} z_n^k = z_k^* := \frac{-\lambda_k}{1-\lambda_k}v_k$ for $k = 1, 2$, and since $\varepsilon_n^0(0) < z_n^0$ and $\bar{\varepsilon}_n(0) \in (z_n^1, z_n^0)$ for all $n$, it follows that $\bar{x}^* \leq z_1^*$ and $\bar{\varepsilon} \geq z_1^*$. To prove Proposition 4, it suffices to show that $\bar{x}^* = \bar{\varepsilon} = z_1^*$.

Suppose by contradiction that $\bar{x}^* < z_1^*$. Let $L^n(x, 0) = E^n[e^{-r\tau_n(0)}(\alpha(P^n(x, \tau_n(0), \alpha) - x, \alpha) + \Pi^n(x, \tau_n(0), \alpha))]$ be the seller’s profits when $\sigma = \sigma_n$, where $\tau_n(0) = \inf\{t : x_t \in [0, \varepsilon_n(0)] \cup [\bar{\varepsilon}_n(0), z_n^0]\}$ and where $E^n[\cdot]$ and $\Pi^n(x, \alpha)$ denote, respectively, the expectation operator and the seller’s profits at state $(x, \alpha)$ when $\sigma = \sigma_n$. Let $\hat{\tau} = \inf\{t : x_t \leq \varepsilon_n^0\}$ and note that $\tau_n(0) \to \hat{\tau}$ as $n \to \infty$ when $x_0 \in (\bar{x}^*, \varepsilon^*)$: as $\sigma \to 0$ the probability that $x_t$ reaches $\bar{x}^*$ before it reaches $\varepsilon^*$ approaches zero if $x_0 \in (\varepsilon^*, \bar{x}^*)$. Moreover, $P^n(x, \alpha) \to v_1$ and $\Pi^n(x, \alpha) \to (1-\alpha)(v_1 - x_*^*)$ as $n \to \infty$. Thus, $\lim_{n \to \infty} L^n(x, 0) = E^n[e^{-r\hat{\tau}}(v_1 - x_*)|x_0 = x]$ for $x \in (\bar{x}^*, \varepsilon^*)$. There are two cases to consider: (i) $\bar{x}^* < z_1^*$, or (ii) $\bar{x}^* > z_1^*$. In case (i), for $n$ large enough the seller’s profits $L^n(x, 0) \approx E^n(e^{-r\hat{\tau}}(v_1 - x_*)|x_0 = x)$ are strictly lower than $v_1 - x_0$ when $x_0 \in (\varepsilon_n^0, z_1^*)$, since, by Lemma 1, $v_1 - x > E^n[e^{-r\tau}(v_1 - x_\tau)|x_0 = x]$ for all $\tau$ and $x < z_1^*$ when $n$ is large enough. This contradicts the fact that $L^n(x, 0) \geq v_1 - x$ for all $x \leq z_1^*$, so $\bar{x}^* = z_1^*$. In case (ii), $L^n(x, 0) \approx E^n[e^{-r\hat{\tau}}(v_1 - x_*)|x_0 = x] \leq (P^n(x, \alpha) - x\alpha + \Pi^n(x, \alpha)$ for all $x \in (z_1^*, \bar{x})$ and for $n$ large enough, since by Lemma 2, $P^n(x, \alpha) - x \geq \sup_n E^n[e^{-r\tau}(v_1 - x_\tau)|x_0 = x]$ and since $P^n(x, \alpha) - (1-\alpha)E^n[e^{-r\tau}(v_1 - x_*)|x_0 = x]$. But this cannot be, since $L^n(x, 0) \geq (P^n(x, \alpha) - x\alpha + \Pi^n(x, \alpha)$. Hence, $\bar{x}^* = z_1^*$.

Proof of Proposition 4. Fix a sequence $(\sigma_n, \mu_n) \to 0$. For each $n$, let $\lambda_n$ be the negative root of $\frac{1}{2}\sigma_n^2\lambda (\lambda - 1) + \mu_n \lambda = r$. For $i = 1, 2$, let $z_n^i = \frac{-\lambda_n}{1-\lambda_n}v_i$. Note that $\lim_{n \to \infty} \lambda_n = -\infty$ and $\lim_{n \to \infty} z_n^i = v_i$. Let $P^n(x, \alpha)$ be buyer $\alpha$’s strategy when $(\sigma, \mu) = (\sigma_n, \mu_n)$. Note that $\lim_{n \to \infty} P^n(x, \alpha) = v_1$ for $x \leq v_1$ and $\lim_{n \to \infty} P^n(x, \alpha) = v_2$ for $x > v_1$. By Theorem 1, for each $n$ there exists $\varepsilon_n^0(0) < z_n^0$ and $\bar{\varepsilon}_n(0) \in (z_n^1, z_n^0)$ such that the monopolist sells at $t = 0$ to all high types at price $P^n(x, \alpha)$ when $x_0 \in [\bar{\varepsilon}_n(0), z_n^0]$, and sells at $t = 0$ to all buyers at price $v_1$ when $x_0 \leq \varepsilon_n^0(0)$. To prove the Proposition it suffices to show that $\bar{x}^* = \lim_{n \to \infty} \varepsilon_n^0(0) = v_1$ and $\bar{\varepsilon} = \lim_{n \to \infty} \bar{\varepsilon}_n(0) = v_1$.

Since $\varepsilon_n^0(0) < z_1^*$ and $\bar{\varepsilon}_n(0) \in (z_1^*, z_2^*)$ for all $n$, it follows that $\bar{x}^* \leq v_1$ and $\bar{\varepsilon} \geq v_1$. Suppose by contradiction that $\bar{x}^* \neq v_1$ or $\bar{\varepsilon} \neq v_1$, so that $\bar{x}^* \neq \bar{\varepsilon}$. Thus, there exists $N$ and $y < \bar{\varepsilon}$ such that $\varepsilon_n^0(0) \leq y$ and $\bar{\varepsilon}_n(0) \geq y$ for all $n \geq N$. Let $L^n(x, 0)$ be the monopolist’s profits at state $(x, 0)$ when $(\sigma, \mu) = (\sigma_n, \mu_n)$. By Theorem 1, $L^n(x, 0) = E^n[e^{-r\tau_n(0)}(\alpha(P^n(x, \tau_n(0), \alpha) - x, \alpha) + \Pi^n(x, \tau_n(0), \alpha))]$, where $\tau_n(0) = \inf\{t : x_t \in [0, \varepsilon_n(0)] \cup [\bar{\varepsilon}_n(0), z_n^0]\}$ and where $E^n[\cdot]$ and $\Pi^n(x, \alpha)$ denote, respectively, the expectation operator and the seller’s profits at state $(x, \alpha)$ when $(\sigma, \mu) = (\sigma_n, \mu_n)$. Let $\hat{\tau} = \inf\{t : x_t \notin (y, \bar{\varepsilon})\}$ and note that for all $n \geq N$, $\tau_n(0) \geq \hat{\tau}$ whenever $x_0 \notin (y, \bar{\varepsilon})$. Fix $x \in (y, \bar{\varepsilon})$. Since $P^n(x, \tau_n(0), \alpha) < v_2$ and $\Pi^n(x, \tau_n(0), \alpha) < (1-\alpha)v_2$ for all $n$, $L^n(x, 0) < v_2E^n[e^{-r\tau}]$ for all $n \geq N$. Finally,
note that \( \lim_{n \to \infty} E^n [e^{-rT}] = 0 \) when \( x_0 \in (y_0, \bar{y}) \): as \((\sigma, \mu) \to 0\) it takes arbitrarily long until costs leave the interval \((y_0, \bar{y})\). This implies that \( L^n(x,0) \to 0 \), which cannot be since \( L^n(x,0) \geq \alpha(P^n(x,\alpha) - x) + (1-\alpha)\Pi^n(x,\alpha) > 0 \) for all \( x < z^n_2 \). Thus, \( \bar{x} = \bar{x}^\ast = v_1 \).

### B.5 Proof of Theorem 2

Theorem 2 generalizes Theorem 1. Here I provide a sketch of the main arguments. Suppose that there are \( n \geq 3 \) types of buyers, with valuations \( v_1 < \ldots < v_n \). For \( k = 1, \ldots, n \), let \( \alpha_k = \max \{ i \in [0,1] : f(i) = v_k \} \) be the highest indexed buyer with valuation \( v_k \). Let \( \alpha_{n+1} = 0 \). For \( q \in [\alpha_3, \alpha_2) \), the only buyers in the market are those with valuations \( v_1 \) and \( v_2 \). By Theorem 1, the seller’s profits are \( L(x,q) = \sup_x E[e^{-rt}((P(x,\alpha_3) - x)(\alpha_2 - q) + (1-\alpha_2)V_1(x_\tau)|x_0 = x) \right. \) for all states \((x,q)\) with \( q \in [\alpha_3, \alpha_2) \). For \( q \geq \alpha_2 \), the seller’s profits are \( L(x,q) = (1-q)V_1(x) \).

Consider next states \((x,q)\) with \( q \in [\alpha_3, \alpha_2) \), at which there are \( \alpha_3 - q \) buyers with valuation \( v_3 \) in the market. By equation (5), the strategy \( P(x,\alpha_3) \) of consumer \( \alpha_3 \) (the highest indexed consumer with valuation \( v_3 \)) satisfies \( P(x,\alpha_3) = v_3 - E[e^{-r\tau_2}(v_3 - P(x_\tau_2,q_\tau_2))|x] \), where \( \tau_2 = \inf \{ t : x_t \leq z_2 \} \) is the time at which the monopolist starts selling to buyers with valuation \( v_2 \) when the level of market penetration is \( \alpha_3 \). By the skimming property, the monopolist can always sell to all buyers with valuation \( v_3 \) at price \( P(x,\alpha_3) \). Therefore, at states \((x,q)\) with \( q \in [\alpha_3, \alpha_2) \) the monopolist’s profits are bounded below by \( L(x,q) = \sup_x E[e^{-rt}((P(x,\alpha_3) - x)(\alpha_3 - q) + e^{-r\tau_2}L(x,\alpha_3)|x_0 = x) \right. \). Using arguments similar to those in Lemma B4, one can show that the solution to this optimal stopping problem is of the form \( \tau(q) = \inf \{ t : x_t \in [0,\bar{x}_1(q)] \cup [\bar{x}_1(q),\bar{x}_2(q)] \cup [\bar{x}_2(q),z_3] \} \), with \( \bar{x}_1(q), \bar{x}_1(q), \bar{x}_2(q), \bar{x}_2(q) \) such that \( \bar{x}_1(q) < z_1 < \bar{x}_1(q) \) and \( \bar{x}_2(q) < z_2 < \bar{x}_2(q) < z_3 \). Moreover, by arguments similar to those in Lemma B5, \( L(x,q) \in C^{2,2} \) for all \( x \in (\bar{x}_1(q),\bar{x}_1(q)) \cup (\bar{x}_2(q),\bar{x}_2(q)) \).

Next, by arguments similar to those in Lemma B7 the monopolist always sells at a continuous rate (i.e., \( dq_t > 0 \)) at states \((x_t,q_t^-)\) with \( x_t \leq z_3 \) and \( q_t^- \in [\alpha_3, \alpha_2) \), and never sells at states with \( x_t > z_3 \) and \( q_t^- \in [\alpha_4, \alpha_3) \). Moreover, arguments similar to those in Lemma B10 imply that in any equilibrium, \( P(x_s,q_s) = x_s - \Pi_q(x_s,q_s) \) whenever the monopolist is selling at a continuous rate (i.e., whenever \( \{q_t\} \) is continuous and strictly increasing). Finally, by arguments similar to those in Lemma B11, in any equilibrium the monopolist’s profits must be equal to \( L(x,q) \) at all states \((x,q)\) with \( q \in [\alpha_3, \alpha_2) \).

At states \((x_t,q_t^-)\) with \( q_t^- \in [\alpha_4, \alpha_3) \) the equilibrium dynamics are as follows. If \( x_t > z_3 \), the monopolist doesn’t sell and waits for costs to decrease. When \( x_t \in [0,\bar{x}_1(q_t^-)] \cup [\bar{x}_1(q_t^-),\bar{x}_2(q_t^-)] \cup [\bar{x}_2(q_t^-),z_3] \) (i.e., when \( x_t \) is in the stopping region of (A.2), the monopolist sells immediately to all remaining consumers with valuation \( v_3 \) at price \( P(x_t,q_t^-) = x_t - L_q(x_t,q_t^-) \)), and then equilibrium play continues as in the case with two types of consumers. Finally, when \( x_t \in (\bar{x}_1(q_t^-),\bar{x}_1(q_t^-)) \cup (\bar{x}_2(q_t^-),\bar{x}_2(q_t^-)) \), the monopolist sells gradually to consumers with valuation \( v_3 \) at price \( P(x_t,q_t^-) = x_t - L_q(x_t,q_t^-) \). In this region, prices evolve in such a way that buyers with valuation \( v_3 \) are indifferent between trading at \( t \) or waiting.

Consider next states \((x,q)\) with \( q \in [\alpha_5, \alpha_4) \), at which there are \( \alpha_4 - q \) buyers with valuation \( v_4 \) in the market. Let \( P(x,\alpha_4) \) be the strategy of consumer \( \alpha_4 \) (i.e., the highest indexed buyer with valuation \( v_4 \)). Equation (5) implies that \( P(x,\alpha_4) = v_4 - E[e^{-r\tau_3}(v_4 - P(x_{\tau_3},q_{\tau_3}))|x] \), where \( \tau_3 = \inf \{ t : x_t \leq z_3 \} \) is the time at which the monopolist sells to buyers with
valuation \( v_3 \) when \( q = \alpha_4 \). Since the monopolist can sell to all buyers with valuation \( v_4 \) at price \( P(x,\alpha_4) \), at states \((x,q)\) with \( q \in [\alpha_5, \alpha_4) \) her profits are bounded below by

\[
L(x,q) = \sup_{\tau \in T} E[e^{-r\tau} g(x,\tau, q)]
\]

where \( g(x,\tau, q) = (P(x,\alpha_4) - x_\tau)(\alpha_4 - q) + L(x,\alpha_4) \). Repeating the same arguments, in equilibrium the seller’s profits are \( L(x,q) \) for \((x,q)\) with \( q \in [\alpha_5, \alpha_4) \). More generally, for \( k \geq 5 \) I can extend \( L(x,q) \) for all \( q \in [\alpha_{k+1}, \alpha_k) \) in a similar way, and show that in equilibrium the seller’s profits are \( L(x,q) \) for all \( q \in [\alpha_{k+1}, \alpha_k) \).

**B.6 The discrete-time game**

This section studies the discrete time version of the model in the paper. The main goal is to show that in any subgame perfect equilibrium of this game, the strategies of the buyers must satisfy condition (iii) in Definition 1 in the main text. For simplicity, I focus on the case in which there are two types of buyers, as in Section 4. I stress however that these results generalize to settings with any (finite) number of types.

As in the main text, a monopolist faces a continuum of consumers indexed by \( i \in [0,1] \). For each \( i \in [0,1] \), let \( f(i) \) denote the valuation of consumer \( i \). There are two types of buyers: high types with valuation \( v_2 \), and low types with valuation \( v_1 \in (0,v_2) \). Let \( \alpha \in (0,1) \) be the fraction of high types in the market, so \( f(i) = v_2 \) for all \( i \in [0,\alpha] \) and \( f(i) = v_1 \) for all \( i \in (\alpha,1) \).

Time is discrete. Let \( T(\Delta) = 0, \Delta, 2\Delta, \ldots \) be the set of times at which players take actions, with \( \Delta \) measuring the time period. At each time \( t \in T(\Delta) \) the monopolist announces a price \( p \). All consumers who have not purchased already simultaneously choose whether to buy at this price or wait. All players have perfect recall of the history of the game. Moreover, all players in the game are expected utility maximizers, and have a common discount factor \( \delta = e^{-r\Delta} \). The monopolist’s marginal cost of production evolves as (1), with \( |\mu| < r \) and \( \sigma > 0 \). The seller’s cost is publicly observable. Note that costs evolve continuously over time, but the monopolist can only announce a price and make sales at times \( t \in T(\Delta) \). Therefore, as \( \Delta \to 0 \), costs become more persistent across periods.

A strategy for the monopolist specifies at each time \( t \in T(\Delta) \) a price to charge as a function of the history. A strategy for a consumer specifies at each time the set of prices she will accept (provided she has not previously made a purchase). I focus on the subgame perfect equilibria (SPE) of this game.\(^3\) \(^4\)

**Lemma B12** In any SPE and after any history, all buyers accept a price equal to \( v_1 \), regardless of the current level of costs.

**Proof.** Fix a SPE and let \( p(x) \) be the supremum of prices accepted by all consumers after any history such that current costs are \( x \). Let \( \underline{p} := \inf_{x \in \mathbb{R}_+} p(x) \). Note first that \( \underline{p} \leq v_1 \), since buyers with valuation \( v_1 \) never accept a price larger than their valuation. Suppose by contradiction that the Lemma is not true, so \( \underline{p} < v_1 \). Note that the monopolist would never

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\(^3\)As usual in durable goods monopoly games, I restrict attention to SPE in which actions are constant on histories in which prices are the same and the sets of agents accepting at each point in time differ by sets of measure zero; see Gul, Sonnenschein and Wilson (1986) for a discussion of this assumption.

\(^4\)Existence of SPE can be shown by generalizing arguments in Gul, Sonnenschein and Wilson (1986).
charge a price lower than \( p \). Consider the offer \( p = (1 - \delta)v_1 + \delta \bar{p} > \bar{p} \). Note that every buyer would accept a price of \( p - \epsilon \) for any \( \epsilon > 0 \), since the price in the future will never be lower than \( p \). Moreover, \( p - \epsilon > p \) for \( \epsilon \) small enough. This implies that there exists a cost level \( x \) such that \( p - \epsilon > p(x) \), which contradicts the fact that \( p(x) \) is the supremum of prices accepted by all consumers after any history such that current costs are \( x \). Thus, \( p = v_1 \). ■

An immediate Corollary of Lemma B12 is that, in any SPE, consumers with valuation \( v_1 \) accept a price equal to \( v_1 \); that is, condition (4) in the main text holds in any SPE. The next result shows that condition (5) also holds in any SPE of the game.

Consider the optimal stopping problem \( \sup_{\tau \in T(\Delta)} E[e^{-r\tau}(v_1 - x_\tau)|x_0 = x] \), where \( T(\Delta) \) is the set of stopping times taking values on \( T(\Delta) \). The solution to this problem is to stop the first time costs fall below some level \( z_1^\Delta \). For all \( s \in T(\Delta) \), let \( \tau_1^\Delta(s) = \inf\{t \in T(\Delta), t > s : x_t \leq z_1^\Delta \} \). Let \( \tau_1^\Delta = \tau_1^\Delta(0) \). Note that, in any SPE, if all high type consumers buy at time \( s \in T(\Delta) \) and leave the market, the monopolist will then wait until time \( \tau_1^\Delta(s) \) and charge a price of \( v_1 \) (which all low type buyers accept). For all \( x > 0 \), let \( P^\Delta(x) = v_1 - E[e^{-r\tau_1^\Delta}(v_1 - v_1)|x_0 = x] \).

**Lemma B13** In any SPE and after any history, all buyers with valuation \( v_2 \) accept a price equal to \( P^\Delta(x) \) if the current cost level is \( x \).

**Proof.** Fix a SPE and let \( p_2(x) \) be the supremum of the prices that all buyers \( i \in [0, \alpha] \) accept after any history if current costs are \( x \). I first show that \( p_2(x) \leq P^\Delta(x) \). To see this, note that by definition of \( p_2(x) \), all buyers \( i \in [0, \alpha] \) that remain in the market will buy if the seller charges a price \( p_2(x) \). By our discussion above, the monopolist will then sell to all low types at a price \( v_1 \) the first time costs fall below \( z_1^\Delta \). The utility that a high type buyer gets by not purchasing at price \( p_2(x) \) and waiting until the monopolist serves low types is \( E[e^{-r\tau_1^\Delta}(v_1 - v_1)|x_0 = x] \). Therefore, for all buyers to be willing to purchase at price \( p_2(x) \), it must be that \( v_2 - p_2(x) \geq E[e^{-r\tau_1^\Delta}(v_2 - v_1)|x_0 = x] \), or \( p_2(x) \leq P^\Delta(x) \).

I now complete the proof by showing that that \( p_2(x) \geq P^\Delta(x) \). To see this, let \( \bar{p}(x) = p_2(x) \) if \( x > z_1^\Delta \) and \( \bar{p}(x) = v_1 \) if \( x \leq z_1^\Delta \). Note that the monopolist will never charge a price lower than \( p(x) \) if current costs are \( x \). Moreover, all high type consumers will accept this price. As a first step to prove the inequality, I show that \( p_2(x) \geq v_2 - E[e^{-r\Delta}(v_2 - \bar{p}(x_{t+\Delta}))|x_t = x] \). To see this, suppose by contradiction that this is not true. Then, some high type consumers would reject a price of \( p'(x) = v_2 - E[e^{-r\Delta}(v_2 - \bar{p}(x_{t+\Delta}))|x_t = x] - \epsilon \) for \( \epsilon \) small enough (i.e., \( p'(x) > p_2(x) \) for \( \epsilon \) small enough). Note that the lowest possible price that the seller would charge next period is \( \bar{p}(x_{t+\Delta}) \), and that this price would be accepted by all high types. This implies that the continuation utility of high types from rejecting today’s price is bounded above by \( E[e^{-r\Delta}(v_2 - \bar{p}(x_{t+\Delta}))|x_t = x] \). But this in turn implies that all high type consumers should accept a price of \( v_2 - E[e^{-r\Delta}(v_2 - \bar{p}(x_{t+\Delta}))|x_t = x] - \epsilon > \bar{p}(x) \), a contradiction to the fact that \( p_2(x) \) is the supremum over all prices that all high types accept when costs are equal to \( x \). Hence, \( p_2(x) \geq v_1 - E[e^{-r\Delta}(v_2 - \bar{p}(x_{t+\Delta}))|x_t = x] \).

For all \( t \in T(\Delta) \), let \( F^\Delta(t, x) = \text{Prob}(\tau_1^\Delta = t|x_0 = x) \). Recall that \( \tau_1^\Delta = \tau_1^\Delta(0) = \inf\{t \in \)
\( T(\Delta), t > 0 : x_t \leq z^\Delta_1 \), so \( F^\Delta(0, x) = 0 \). Note then that

\[
p_2(x) \geq v_2 - E[e^{-r\Delta}(v_2 - p(x\Delta))|x_0 = x]
= v_2 - e^{-r\Delta}F^\Delta(\Delta, x)(v_2 - v_1)
- e^{-r\Delta}(1 - F^\Delta(\Delta, x))E[e^{-r\Delta}(v_2 - p_2(x\Delta))|x_0 = x, \tau^\Delta_1 > \Delta],
\]

(B.12)

where the equality follows since \( p(x) = v_1 \) for all \( x \leq z^\Delta_1 \) and \( p(x) = p_2(x) \) for all \( x > z^\Delta_1 \). Using the fact that \( p_2(x) \geq v_2 - E[e^{-r\Delta}(v_2 - p(x\Delta))|x_0 = x] \) repeatedly in equation (B.12), it follows that

\[
p_2(x) \geq v_2 - \sum_{k=1}^{\infty} e^{-rk\Delta}(v_2 - v_1)F^\Delta(k\Delta, x) = v_2 - E[e^{-r\Delta}(v_2 - v_1)|x_0 = x],
\]

where the last equality follows since \( F^\Delta(t, x) = \text{Prob}(\tau^\Delta_1 = t|x_0 = x) \) and since \( F^\Delta(0, x) = 0 \).

Lemma B13 shows that condition (5) holds in any SPE of this discrete time game with two types of buyers: all buyers \( i \in [0, \alpha] \) accept a price that leaves them indifferent between buying at that price or waiting and buying at the time low type consumers buy; in particular, consumer \( \alpha \) accepts such a price.

Lemmas B12 and B13 together establish that condition (iii) in Definition 1 holds in any SPE of this discrete time game with two types of consumers. If there were three types of consumers, with valuations \( v_3 > v_2 > v_1 \), then the monopolist would only serve \( v_2 \)-consumers when costs are below some cutoff \( z^\Delta_2 \). Letting \( P^\Delta_2(x) \) denote the price at which the monopolist first sells to consumers with valuation \( v_2 \) (when costs are \( x \)), arguments identical to those in Lemma B13 can be used to show that, in any SPE, all consumers with valuation \( v_3 \) accept a price equal to \( P^\Delta_3(x) = v_3 - E[e^{-r\Delta^2}(v_3 - P^\Delta_2(x\Delta_2))|x_0 = x] \), where \( \tau^\Delta_2 = \inf\{t \in T(\Delta), t > 0 : x_t \leq z^\Delta_2 \} \). Hence, condition (5) also holds in any SPE of this discrete time game with three types of buyers. Repeating this argument, one can show that condition (5) holds in any SPE of this game with any finite number of consumer types. Moreover, the arguments in Lemma B12 don’t rely on there being only two types of buyers (i.e., the same arguments would apply to settings with any number of consumer types). Therefore, condition (4) would also hold in any SPE of this game with any number of consumer types.

### B.7 Full commitment

In this appendix, I solve for the full commitment strategy of the monopolist when there are two types of buyers in the market. In the full commitment problem, the monopolist chooses a path of prices \( \{p_t\} \). Given a path of prices \( \{p_t\} \), consumer \( i \) makes her purchase at the earliest stopping time that solves \( \sup_r E[e^{-r\tau}(f(i) - p_t)] \). Hence, with two types of buyers, there will be (at most) two times of sale: the (random) time \( \hat{\tau}_1 \) at which low types buy, and the (random) time \( \hat{\tau}_2 \) at which high types buy. Moreover, high types will buy weakly earlier than low types, so \( \hat{\tau}_2 \leq \hat{\tau}_1 \) with probability 1. Note that, by choosing the path of prices, the monopolist effectively chooses the times \( \hat{\tau}_1 \) and \( \hat{\tau}_2 \) at which the different consumers buy.

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\(^5\{p_t\} \) must be a progressively measurable process.
Note next that the monopolist will always charge a price of $v_1$ to low type buyers. Given this, the highest price that the monopolist can charge high type buyers is given by $p(x_t) = v_2 - E[e^{-r(\hat{\tau}_1-t)}(v_2-v_1)|x_t]$. Therefore, the optimal strategy of the monopolist boils down to optimally choosing the times $\hat{\tau}_1$ and $\hat{\tau}_2$ at which the different consumers buy. That is, the monopolist’s full commitment profits $\Pi^{FC}(x)$ are given by

$$\Pi^{FC}(x) = \sup_{\hat{\tau}_1,\hat{\tau}_2} \alpha E\left[e^{-r\hat{\tau}_2}(p(x_{\hat{\tau}_2}) - x_{\hat{\tau}_2}) | x_0 = x\right] + (1 - \alpha) E\left[e^{-r\hat{\tau}_1}(v_1 - x_{\hat{\tau}_1}) | x_0 = x\right],$$

where the equality follows from using $p(x_{\hat{\tau}_2}) = v_2 - E[e^{-r(\hat{\tau}_2-x_{\hat{\tau}_2})}(v_2-v_1)|x_{\hat{\tau}_2}]$ and from applying the law of iterated expectations. Note the solution to the problem above involves choosing $\hat{\tau}_2$ to maximize the first term, and choosing $\hat{\tau}_1$ separately to maximize the second term. Moreover, by Lemma 1, $\hat{\tau}_2 = \tau_2 = \inf\{t: x_t \leq z_2\}$. Finally, note that the second term is always negative if $v_1 \leq \alpha v_2$, so in this case the optimal strategy for the monopolist is to set $\hat{\tau}_1 = \infty$; that is, to never sell to low types. In this case, $\Pi^{FC}(x) = \sup_x \alpha E\left[e^{-r(x)}(v_2 - x) | x_0 = x\right]$. Otherwise, if $v_1 > \alpha v_2$, one can use arguments similar to those in the proof of Lemma 1 to show that it is optimal to set $\hat{\tau}_1 = \inf\{t: x_t \leq -\lambda^{-1} \lambda(v_1 - \alpha v_2)\}$.

References


