Asset Pricing and Bank Lending Equilibrium with Collateral

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Abstract

While much of the existing financial literature on asset pricing and corporate finance assumes an exogenous lending market, this paper studies an endogenous one in a general equilibrium model. To achieve this, a natural instrument is collateral, because its value is associated with the conditional liquidation price. In the economy, all agents are endowed with a collateralizable asset and cash, and some (entrepreneurs) also own private investments. This collateralizable asset can be traded or used as collateral to borrow from banks to finance private investments. The collateral value and amount of borrowing are endogenously determined by all agents' wealth. This endogeneity allows the collateralizable asset price to be less sensitive to the change in the return on these private investments, due to the wealth effect of the private investments on the price, an effect that does not exist in an exogenous lending market. In addition, when entrepreneurs can choose among different private investments, banks facilitate diversification of these choices by subsidizing the less profitable ones. This subsidized loan theory has two advantages: to study explicitly the role of banks in the capital budgeting literature and to shed light on the empirical findings about the negative correlation between leverage and profitability.

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1 Introduction

Conventional asset pricing and corporate finance theories assume a partial equilibrium lending market in which an agent or firm borrows against its own creditworthiness. Specifically, many of them assume agents can lend to each other, as in asset pricing literature, or the lending market is exogenously given and lenders have no active role other than to answer borrowing requests, as in corporate finance literature. In reality, neither of these is true. Banks perform the majority of lending, while actively managing a portfolio of credit risk. The pursuit of managing the portfolio of credit risk raises serious doubts about the assumption that creditworthiness only matters at the individual level, instead, banks’ portfolio view suggests that it should be determined in the aggregate economy, in the same spirit as in Modern Portfolio Theory: the risk comes from variance as well as covariance. Therefore, the lending market assumed in a partial equilibrium model no longer applies to general equilibrium.

The major theories in capital asset pricing—the CAPM—as well as those in corporate finance—the Modigliani and Miller theory, the pecking order theory and the tradeoff theory—are all based on a partial equilibrium lending market. Therefore, it may not be appropriate to apply them to explain the aggregate economy or cross section variations. What we need is a new lending market determined in general equilibrium, taking into consideration all the individuals’ borrowing activities.

To achieve this, we model a lending market in which collateral plays an explicit role. Collateral is used to buffer against contractual defaults and thus is able to shift the credit risk from the single borrower to the collateral value. Depending on whether the lender will liquidate collateral or hold it till maturity, the collateral value
can be determined by the liquidation price or the fundamentals of the underlying asset. On many occasions, lenders carry a cost to maintain collateral, such as in the mortgage industry, or they may fear a further loss, particularly in a crisis, so to liquidate collateral is a common action. The liquidation price is not the unconditional expected selling price, but one conditional on the borrower’s default. Therefore, the liquidation price reflects the covariance between the borrower’s wealth and the aggregate economy. As a result, collateral value is not just a function of the asset, but also of the borrower. A simple haircut defined as the ratio of collateral value to the market price is unable to fully reflect this information.

Moreover, by including collateral, we can study the price impact of an asset after its use as collateral. This is important because there is an increasing number of securities that can serve as collateral due to financial innovations. The recent financial crisis in 2008 has also involved the use of collateral.

We construct a two-period economy with three entities—entrepreneurs, investors and banks—and two endowed assets—cash and a collateralizable asset.

Entrepreneurs are distinguished from investors by their private investments. They have three ways to finance their private investments: endowed cash, proceeds from selling the endowed collateralizable asset to investors, or a loan from a bank which requires this asset as collateral. In the case of borrower default, the bank will liquidate collateral at a price dependent on all the potential buyers’ cash holdings (see Oehmke 2008 for dynamically liquidating collateral). By making the borrowing capacity equal to this liquidation price, the borrowing activities for all agents are simultaneously determined in general equilibrium. We call this an endogenous lending market, in
contrast to the exogenous one determined by the partial equilibrium in which no interaction exists among all the borrowing activities.

This paper proves that this endogenous lending market is indeed different from two commonly-used exogenous ones: the one with borrowing constraints (restricted lending market) and the one without (unlimited lending market). Two implications arise from this difference.

Firstly, in the endogenous lending market, the collateralizable asset price is less sensitive to the change in returns on these private investments in the endogenous lending market than in the restricted one; the reason is that the substitution effect between the asset and private investments can be partially offset by the wealth effect that exists only in the former market.

Secondly, in the endogenous lending market, banks facilitate diversification among entrepreneurs’ choices of private investments. We demonstrate this by giving them two private investment options with different profitabilities. In the existing capital budgeting literature that assumes an exogenous lending market, entrepreneurs always choose the more profitable one. In contrast, in the endogenous lending market, they might choose the less profitable one in equilibrium, due to subsidized loans from banks. Banks have incentives to offer such loans to avoid dealing with entrepreneurs making the same choice, as in the idea of the Modern Portfolio Theory. In an extreme case, consider that the less profitable investment has a return negatively correlated with the aggregate economy. If an entrepreneur borrows to finance this investment, the bank can either obtain a full repayment from the borrower or sell collateral when the aggregate economy is doing well. In either case, the bank faces little risk. Therefore,
the loan could be attractive enough to the borrower to compensate for the reduced profitability. In one sense, the loan from the bank subsidizes the less profitable investment, or put it another way, leverage is essential to carry a less profitable investment. This prediction is in line with the existing literature on firms’ capital structure and profitability (Hail and Weiss 1967 and Gale 1972). In contrast to the pecking order theory that only considers the firm’s perspective, we provide a bank-firm joint analysis.

We also show that banks value collateral the least if they face a pool of identical entrepreneurs, in the same spirit as Shleifer and Vishny (1992). These authors take a game theory approach to endogenize the collateral value. In contrast, we emphasize the role of correlations among agents’ wealth in determining this value.

Our model of collateral equilibrium has its root in a series of papers by Geanakoplos, such as Geanakoplos (1997) and Geanakoplos (2003). Geanakoplos (2003) emphasizes the role of collateral in leverage and asset price crashes, whereas we focus on the role of collateral in forming an endogenous lending market. Moreover, Geanakoplos (2003) attributes the higher price of an collateralizable asset to the excessive demand by the use of leverage, on the other hand, we argue that it comes from the action of using the asset as collateral per se. As long as some entrepreneurs use the asset as collateral to borrow, the price will be higher than in the situation where all entrepreneurs crowd the market to sell. In other words, the action of using the asset as collateral is sufficient to cause a higher price, regardless of the purpose. Therefore, we broaden the scope to explain the higher price caused by this use of collateral.

Previous studies on bank lending and collateral have mainly explored issues in
partial equilibrium, such as asymmetric information between borrowers and lenders (Besanko and Thakor 1987; Bester 1987), and the quality of collateral (Plaut, 1985). A key difference between studies in the partial and general equilibrium lies in the valuation of collateral. In the case of partial equilibrium, the collateral value is exogenously given. Whereas in the general equilibrium, the collateral value is endogenously determined by all the agents.

Bank lending is related to credit constraints in macroeconomics. Bernanke and Gertler (1989), Bernanke, Gertler and Gilchrist (1997) have shown that credits based on a borrower’s balance sheet may have a pro-cyclical effect on the business cycle. Kiyotaki and Moore (1997) show how an exogenous shock to the economy has ripple effects across time, further amplified by the use of collateral. They all highlight the role of banks in exaggerating the business cycle across time, but in the cross-sectional view, our model indicates that banks also tend to attenuate variation by facilitating a diversified economy.

Lastly, this study is related to an extensive literature on banking. Most existing banking theories examine credit channels cross sectionally by focusing on the mechanism through which banks acquire capital from savers and transfer it to lenders. Diamond and Dybvig (1983), Diamond and Rajan (2000), Archaya, Gorton and Metrick (2009) and Gale and Yorulmazer (2010) study the risks when banks operate lending with short term funding. Stiglitz and Weiss (1981) study credit rationing in an asymmetric information environment. In this study, we link the lending market to the future wealth of the aggregate economy via collateral, and examine the role of the credit channel across time.
2 The Model

2.1 The Agents

We construct an economy consisting of two time periods 0 and 1, and a continuum of agents, denoted by agent-\(i\) \(i \in [0, 1]\). The risk free rate is \(r = 1\). All the results hold with a risk free rate greater than one, \(r > 1\). At time 0, for \(i \in [0, 1]\), agent-\(i\) is endowed with \(e_i(0)\) units of a collateralizable asset \(x\) and \(e_i(1)\) units of cash, denoted by \(\tilde{e}_i = (e_i(0), e_i(1))\). \(x\) can only be traded at time 0 in a competitive market where all agents are price takers. At time 1, \(x\) generates a bounded positive random payoff \(x \in [x_{min}, x_{max}]\).

For a fraction of agents indexed by \([0, \delta]\), assume each also owns a private investment available only to himself, denoted by \(y_i\) for agent-\(i\). Those agents are called entrepreneurs. The others in \((\delta, 1]\) are called investors. If entrepreneur-\(i\) spends cash \(c\) on private investment \(y_i\), he will obtain \(cy_i\) at time 1 where \(y_i \in R^+\) is a random return. All \(y_i\)s are independent of \(x\) and identically distributed with a distribution function \(\tilde{y}\). Consider two cases: all \(y_i\)s are independent and all \(y_i\)s are identical. Denote by \(\Phi_{in} = \{y_i | i \in [0, \delta]\text{ and } y_i \text{s are independent.}\}\) and \(\Phi_{id} = \{y_i | i \in [0, \delta]\text{ and } y_i \text{s are identical.}\}\)

Assume all entrepreneurs are symmetric, meaning, they have the same utility function and endowments, denoted by \(u\) and \((e(0), e(1))\), respectively. To rule out corner solutions, assume investors have a sufficient amount of cash to buy all \(x\) from entrepreneurs, that is,

**Condition 1 (Sufficient Cash)** \(\int_0^\delta e_i(1)di > \mu_x \int_{0}^\delta e_i(0)di,\)
where $\mu_x = \mathbb{E}x$ is the expected payoff of $x$.

All agents maximize the expected utility of their final wealth at time 1. To uniform notations, define $y_i = 0$ for investors $i \in (\delta, 1]$. Denote by $u_i$ agent-$i$’s utility function satisfying

(i) $u' > 0$, $u'' \leq 0$,

(ii) $u''$ is continuous.

At time 0, agent-$i$ chooses a wealth portfolio $w_i = (a_i, b_i, c_i)$ to maximize

$$Eu_i(a_i x + b_i + c_i y_i)$$ (1)

subject to

(i) $a_i p + b_i + c_i = e_i(0)p + e_i(1)$,

(ii) $a_i \geq 0$, $b_i \geq 0$ and $c_i \geq 0$,

where $a_i$ is the asset $x$ position, $b_i$ is the cash position, $c_i$ is the security $y_i$ position and $p$ is the market price of $x$. Short sale of asset $x$ is not allowed, for short sale is a liability to short sellers and such liability cannot be enforced in a weak enforcement environment. The optimal demand function for $x$ is denoted by $\bar{a}_i$. $\bar{a}_i$ is a function of the endowments $(e_i(0), e_i(1))$ and the price $p$. Write $\bar{a}_i$ as $\bar{a}_i(p, e_i(0), e_i(1))$.

For investors, assume as a group they are willing to hold all $x$ if the price $p$ is small enough:

$$\lim_{p \to 0} \int_0^1 \bar{a}_i(p, e_i(0), e_i(1)) di > \int_0^1 e_i(0) di.$$ (2)

The definition of the market equilibrium is as follows.

**Definition 1 (Market Equilibrium)** At time 0, the market for $x$ is in equilibrium
if the following conditions are satisfied:

(1) Both entrepreneurs and investors maximize their utilities;

(2) The market for \( x \) clears.

Because entrepreneurs are symmetric, they have the same optimal demand function for \( x \) and invest the same cash in security \( y_i \)s.

The primary idea of this study is to compare the different means for entrepreneurs to raise cash for their private investments: selling asset \( x \) or use it as collateral to borrow. If both entrepreneurs and investors start with suboptimal wealth portfolios, they may still want to trade in the market just to rebalance their portfolios, even without any private investment. To focus on the entrepreneurs’ trading incentives for raising cash for their private investments, the following starting equilibrium condition is imposed. The condition stipulates that there is no need for all the agents to trade in the market, if entrepreneurs have no private investments.

**Condition 2 (Starting Equilibrium)** Without private investment \( y_i \)s, all the agents are in equilibrium with their endowments. In other words, there exists a price \( p^* \) such that, for \( i \in [0, 1] \), agent-\( i \)’s endowments \((e_i(0), e_i(1))\) maximize

\[
Eu_i(ax + b)
\]

subject to

(i) \( ap^* + b = e_i(0)p^* + e_i(1) \),

(ii) \( a \geq 0, \text{ and } b \geq 0 \).
The next condition is to make sure the trades flow one direction: it’s the entrepreneurs that sell asset $x$. Now, entrepreneurs have to compare whether to sell $x$ or use $x$ as collateral to borrow.

**Condition 3 (Capital Competing)** Entrepreneurs demand less $x$ after they have $y_i$s, that is, for all $i \in [0, \delta]$,

$$a_i^*(p, e_i(0), e_i(1)) > \bar{a}_i(p, e_i(0), e_i(1)), \quad (4)$$

where $a_i^*$ is the optimal holding of $x$ for entrepreneur-$i$ in condition 2. In a sense, the private investment $y_i$s compete with $x$ for capital.

### 2.2 The Banks

Assume banks have perfect information on each agent. Banks only offer a discount loan. This discount loan is one on which the interest is deducted from the face amount when the loan is offered. The borrower only receives the principal after the interest is deducted but must repay the full amount of the loan. Assume $R$ is the interest rate charged by banks. Further assume that the loan is nonrecourse and the law to enforce repayment is weak. Therefore collateral is the only instrument for banks to protect against loan losses. To be specific, when a borrower defaults, the bank can only seize the collateral, but has no right to claim the borrower’s wealth beyond that. Therefore, the value of collateral has to be large enough to fully cover loan losses. If the loan principle is $l$, the value of collateral has to be greater than or equal to $l$.

Assume there are a sufficient number of banks in the competitive lending business
and each is endowed with sufficient cash as capital. Competition implies that banks demand collateral worth the same as the loan principle. Moreover, given that the loan losses have been fully covered by collateral, banks must earn zero profits on the repayment, charging the risk free rate for the loan, that is, $R = r = 1$.

After banks seize the collateral, assume they will liquidate it immediately, instead of holding it till maturity. There are two major reasons for banks to do so: maintenance costs and further deterioration of the collateral value. The repayment date must be before the payoff of $x$. In addition, this timing makes it impossible for all agents to finance the purchasing of $x$; this confines the use of borrowed cash solely on $y_i$, for $i \in [0, \delta]$, and makes entrepreneurs the only borrowers.

At time 0, entrepreneurs use $x$ as collateral to borrow from banks, and repay loans at time 1 between the realization of $y$ and $x$. The sequence of time 1 events is illustrated in figure 1.

The lending policy is summarized as the following rule.

**Rule 1 (Lending Policy)** For a loan with principle $l$, banks require the same value $l$ of asset $x$ as collateral and charge a risk free rate $R = 1$. This is equivalent to assuming that banks have zero Value at Risk in a competitive lending business.

Since it does not matter which bank an entrepreneur borrows from, we can view
all the banks together as one aggregate bank, or the bank. It is sufficient to study
the behavior of this bank that operates according to the lending policy described in
rule 1.

A. Entrepreneurs’ optimal borrowing

When the bank calculates the collateral value, it needs to estimate the total quan-
tity of collateral to be liquidated in the market at time 1 when borrowers default.
Therefore the bank has to study the borrowers’ repayment behavior. To keep the
model simple, assume there is no renegotiation between entrepreneurs and the bank
once entrepreneurs default on the loan. Entrepreneurs can choose to either repay or
default on the entire loan. All the result in this section hold if the model is extended
to a simple type of renegotiation: partial repayment$^1$.

Denote by $v(\beta)$ the collateral value for $\beta$ units of asset $x$ calculated by the bank,
for $\beta \in [0,1]$. For entrepreneur-$i \in [0, \delta]$, assume he uses $\beta_i$ units of $x$ as collateral to
borrow $\beta_i v(\beta_i)$ at time 0. Assume he has also used endowed cash $e(1) - b_i$ for $y_i$. At
time 1 after the payoff of $y_i$, he decides whether to repay the loan, that is, he chooses
1$_F$ to maximize

$$Eu_i((e_i(0) - \beta_i)x + (e(1) - b_i + \beta_i v(\beta_i))y_i + b_i + 1_{F}(\beta_i x - \beta_i v(\beta_i)))$$ (5)

subject to

$^1$The partial repayment rule is:

**Rule 2 (Partial Repayment Rule)** When repaying the loan, entrepreneurs can repay a fraction
of it to redeem the collateral at the same ratio, that is, repay $\kappa l$ to redeem $\kappa l \beta$ units of $x$, where
$\kappa \in [0,1]$. 
(i) $1_{F_i} \in \{0, 1\}$, and
(ii) $(e(1) - b_i + \beta_i v(\beta_i))y_i + b_i - 1_{F_i}\beta_i v(\beta_i) \geq 0$.

$1_{F_i} = 1$ means entrepreneur-i repays the loan fully and $1_{F_i} = 0$ means he defaults on the loan completely. $1_{F_i}$ is a function of $y_i$, written as $1_{F_i}(y_i)$. By symmetry, in equilibrium, all borrowers hold the same portfolio at time 0 and hence have the same repayment function. The subscript “i” in $1_{F_i}$ can be dropped. Rewrite it as $1_{F}(y_i)$. Denote by $D = \{d|1_{F}(d) = 0\}$ the set of returns for $y$ causing the borrowers to default. In other words, the bank is to liquidate collateral when the borrowers have returns from $y_i$s in the set $D$. Without further regulations on the entrepreneurs’ utility function $u$, it may always be optimal for them to default on the loan. Two conditions regarding the positive wealth effect and the downward sloping demand curve are required to discipline entrepreneurs’ repayment behaviors. Entrepreneurs’ willingness to repay the loan is consistent with that in a repeated borrowing environment in which their reputation of repaying the loan is considered by a bank as an significant factor.

**Condition 4 (Positive Wealth Effect)** For the demand function $a^*_i$ in condition 2,

$$
\frac{\partial a^*_i}{\partial e_i(1)} \geq 0 \quad (6)
$$

holds for all $i \in [0, 1]$.

**Condition 5 (Downward Sloping Demand Curves)** The demand function $a^*_i$ in condition 2 is a decreasing function of price $p$, $\frac{\partial a^*_i}{\partial p} \leq 0$, for all $i \in [0, 1]$.

Given these conditions, the entrepreneurs’ repayment strategy is summarized as a proposition.
**Proposition 1 (Optimal Repayment)** There exists a critical point $y^*$ such that an entrepreneur is willing to repay the loan if $y \geq y^*$ and default completely if $y < y^*$, and another $y^{**}$ such that he has the ability to repay the loan if $y \geq y^{**}$. To sum up, $D = [y_{min}, y^* \lor y^{**}]$.

After solving $1_F(y_i)$, the entrepreneur-i at time 0 chooses a wealth portfolio $(\beta_i, b_i)$ to maximize

$$Eu((e(0) - \beta_i)x + (e(1) - b_i + \beta_i v(\beta_i))y_i + b_i + 1_F(y_i)(\beta_i x - \beta_i v(\beta_i))) \quad (7)$$

subject to
(i) $0 \leq \beta_i \leq e(0)$ and
(ii) $0 \leq b_i \leq e(1)$.

**B. The bank’s collateral valuation**

This bank assesses the collateral, not by its fundamental value, but by market value. Fundamental value is the utility obtained from consuming or holding the asset. The market value is how much one receives when selling it in the market. The bank can only seize and liquidate collateral when their borrowers default at time 1. Since the market is closed at time 1, the bank sells $x$ to all the agents over the counter. In fact, it quotes an asking price for them to purchase.

The way for the bank to compute the collateral value is ”guess and verify later”. The bank guesses in equilibrium, entrepreneurs in $[0, \delta_m]$ transact in the market, and those in $(\delta_m, \delta]$ borrow with collateral, in which $0 \leq \delta_m \leq \delta$. In the market, according to the symmetric maximization problem (1), each entrepreneur in equilibrium has
the same wealth portfolio, denoted by \( w_{\delta_m} = (a_{\delta_m}, b_{\delta_m}, c_{\delta_m}) \) as a function of \( \delta_m \). The equilibrium price and investor-i’s wealth portfolio are denoted by \( p_{\delta_m} \) and \( w_{i\delta_m} = (a_{i\delta_m}, b_{i\delta_m}, c_{i\delta_m}) \), respectively. It can be seen that, in equilibrium, the wealth portfolios for all agents in \([0, \delta_m] \cup (\delta, 1]\) are solely determined by the variable \( \delta_m \).

Entrepreneurs in \((\delta_m, \delta]\) choose to borrow from the bank. By symmetry, in equilibrium, they use the same quantity of \( x \) as collateral to borrow the same amount of cash from the bank. Assume each uses \( \beta \) units of collateral and the collateral value computed by the bank is \( v(\beta) \). The valuation depends on the correlations among \( y_i \)s. The difference can be seen from the following two special cases.

**Case 1: all \( y_i \)s are independent**

Because there are a continuum of entrepreneurs in \((\delta_m, \delta]\), the measure of defaulting entrepreneurs is exactly \((\delta - \delta_m) \Pr(\tilde{y} \in D)\). At time 1, there’ll be exactly \((\delta - \delta_m)\beta \Pr(\tilde{y} \in D)\) units of \( x \) to be liquidated by the bank. The wealth portfolios for all borrowers are functions of \( \delta_m \) and \( v(\beta) \). In the market, the equilibrium portfolios for all market participants are functions of \( \delta_m \). Define a set including all agents’ wealth information at time 1, \( h(\delta_m, \beta, v(\beta)) = h_{[0, \delta_m]} \cup h_{(\delta_m, \delta]} \cup h_{(\delta, 1]} \) where \( h_{[0, \delta_m]} \), \( h_{(\delta_m, \delta]} \) and \( h_{(\delta, 1]} \) are the wealth information sets for the entrepreneurs in the market, the entrepreneurs with the bank and the investors, respectively. They are

\[
\begin{align*}
    h_{[0, \delta_m]} & = \{ w_{1i} = (a_{\delta_m}, ((e(0) - a_{\delta_m})p_{\delta_m} + e(1) - c_{\delta_m})y_i + c_{\delta_m})| i \in [0, \delta_m) \}, \\
    h_{(\delta_m, \delta]} & = \{ w_{1i} = (e(0) - \beta + \beta 1_F(y_i), (e(1) - b_i + \beta v(\beta))y_i \\
    & \quad + b_i - \beta v(\beta) 1_F(y_i))| i \in (\delta_m, \delta]\} \text{ and} \\
    h_{(\delta, 1]} & = \{ w_{1i} = (a_{i\delta_m}, b_{i\delta_m})| i \in (\delta, 1]\}.
\end{align*}
\]
Seen from the continuum of entrepreneurs, both \( h_{(0,\delta_m)} \) and \( h_{(\delta_m,\delta]} \) do not vary with the random returns \( y_i \)s and thus are fixed. The wealth sets for agents in the market, \( h_{[0,\delta_m]} \) and \( h_{(\delta,1]} \), are determined solely by \( \delta_m \) while the set \( h_{(\delta_m,\delta]} \) is determined by both \( v(\beta) \) and \( \delta_m \).

With the wealth information set \( h \) for all agents, the demand function for \( x \) can be derived. The market is closed at time 1, so the bank sells the collateral by quoting an ask price. Observing the price, all agents can decide how much to buy, but cannot sell. Specifically, assume an agent with wealth portfolio \( w_1 = (a_1, b_1) \) observes the ask price \( p \), where \( a_1 \) is \( x \) position and \( b_1 \) cash position. The agent maximizes

\[
Eu(ax + b)
\]

subject to

(i) \( ap + b = a_1 p + b_1 \), and

(ii) \( a \geq a_1 \).

The second constraint requires the agent to purchase only \( x \). The optimal demand for \( x \) is denoted by \( a'(p) \). This agent then buys from the bank \( max(0, a'(p) - a_1) \) units of \( x \) at the quoted price \( p \). Denote such a demand by \( D_i(w_{1i}, p) \) for agent-\( i \) where \( w_{1i} \) is agent-i’s wealth portfolio at time 1. The aggregated purchase from all agents is \( D(h, p) = \int_0^1 D_i(w_{1i}, p) \). The collateral value \( v(\beta) \) solves

\[
D(h, v(\beta)) = (\delta - \delta_m) \beta \Pr(\bar{y} \in D).
\]

The existence of \( v(\beta) \) is guaranteed by the continuity of equation (9). The right hand is a constant number. For the left hand, if \( v(\beta) \) is sufficiently small, the total
purchasing from investors alone could exceed the right hand side according to equation (2). As $v(\beta)$ approaches $x_{\text{max}}$, the total purchasing approaches zero. So, there must exist a solution for $v(\beta)$.

If there are several solutions for $v(\beta)$, in equilibrium, only the one with the highest value will be favored by borrowers. Driven by competition, the bank offers the highest value possible for $v(\beta)$ in equilibrium. Rewrite $v(\beta)$ as $v_{\delta_m}(\beta, \Phi_{\text{in}})$, emphasizing it’s a function of $\delta_m$ and $\Phi_{\text{in}}$, the fraction of entrepreneurs in the market and the private investment returns, respectively.

**Case 2: all $y_i$ are identical**

Now if a borrower defaults, all borrowers default because they hold the same portfolio at time 0 and obtain the same return from securities $y_i$s. The zero Value-at-Risk constraint stipulates that the bank evaluates the collateral in the worst scenario, that is, all $y_i$s generate the lowest return in $D$, denoted by $y_{\text{min}}$. The amount of collateral to be liquidated by the bank is $(\delta - \delta_m)\beta$. The set of wealth information $\mathcal{H}$ is the same as before except now $y_i = y_{\text{min}}$ for all entrepreneurs. As before, write $v(\beta)$ as $v_{\delta_m}(\beta, \Phi_{\text{id}})$. It immediately follows that $v_{\delta_m}(\beta, \Phi_{\text{in}}) > v_{\delta_m}(\beta, \Phi_{\text{id}})$, for there is more collateral to be liquidated when the aggregate economy has minimum wealth.

**Proposition 2 (Collateral Value Comparison)** *Given $0 \leq \delta_m \leq \delta$ and $0 \leq \beta \leq e(0)$, it always holds that $v_{\delta_m}(\beta, \Phi_{\text{in}}) > v_{\delta_m}(\beta, \Phi_{\text{id}})$. The collateral value is higher in a diversified economy in which entrepreneurs have independent private investments.*
2.3 Summary

As the collateral value is determined by the wealth of the aggregate economy conditional on borrowers’ default, the correlation among the wealth of agents is an important factor. As shown in the proposition, the collateral value tends to be higher in a diversified economy.

2.4 The Bank and Market Equilibrium

Now consider an economy with both the bank and a market. To finance $y$, entrepreneur $i$ has two choices: to sell $x$ in the market or to use $x$ as collateral to borrow from the bank. In a competitive market, he is only allowed to submit the demand curve to the Walrasian auctioneer who then determines the price. With this price, the entrepreneur learns the utility obtained in the market. On the other hand, the bank announces the loan terms at the beginning, namely, the collateral value function $v(\beta)$. Observing $v(\beta)$, the entrepreneur is able to compute the maximum utility he can achieve before actually borrowing from the bank.

There is no mechanism for an entrepreneur to make decisions by taking into consideration both the market and the bank at the same time, considering he does not know the exact amount of utilities he can obtain from the market. It is natural to think of his decision as a sequence. He first computes the maximum utility from the bank. This utility is his reservation utility for his next step to participate in the market. For a given price in the market, he can compute this utility. If this utility is greater than the reservation utility, he will submit the quantity together with the price to the auctioneer; otherwise, he will not reveal his demand to the auctioneer.
at that particular price. Therefore, for entrepreneurs, the auctioneer may only have partial demand curves. But for investors, they submit the normal continuous demand curves, for they do not have any reservation utility to participate in the market. The example of a partially revealing demand can be seen in figure 2.

By symmetry, all entrepreneurs have the same reservation utility from borrowing, denoted by \( \pi_b \); they must submit the same demand to the auctioneer. In maximization problem (1). Denote by \( P(\pi_b) \) the set of prices where entrepreneurs obtain a higher utility than \( \pi_b \), that is,

\[
P(\pi_b) = \{ p | Eu(\bar{a}(p, e(0), e(1)))x + e(1) + e(0)p - \bar{a}(p, e(0), e(1))p \geq \pi_b \}.
\]  

Entrepeneurs submit to the auctioneer demands \( \{(p, \bar{a}(p, e(0), e(1)))| p \in P(\pi_b) \} \).

Because not all demands are continuous, the auctioneer may not be able to clear the market for all entrepreneurs and investors. According to condition 3, entrepreneurs should be sellers of \( x \) in equilibrium. If entrepreneurs obtain a utility higher than \( \pi_b \) for price \( p_1 \in P(\pi_b) \), the prices are higher than \( p_1 \), because for the higher price, they can sell less \( x \) to raise the same amount of cash in \( y \). Define \( p(\pi_b) \) such that \( p(\pi_b) = \inf P(\pi_b) \), namely, entrepreneurs prefer to sell the asset in the market for prices greater than or equal to \( p(\pi_b) \).

Revisit the market with entrepreneurs in \([0, \delta_m]\) and all investors in \((\delta, 1]\), as guessed by the bank in the previous section. If the equilibrium price \( p_{\delta_m} \) is greater than or equal to \( p(\pi_b) \), the equilibrium will not be affected by the fraction of demands submitted by those entrepreneurs.

For those \( \delta_m \) such that \( p_{\delta_m} \geq p(\pi_b) \), the auctioneer is able to clear the market for
entrepreneurs \([0, \delta_m]\) and investors \((\delta, 1]\). Since there are more than one ways to clear the market, the auctioneer is required to clear the market for as many entrepreneurs as possible. Specifically, the auctioneer clears the market for all investors and entrepreneurs \([0, \delta_m]\) where \(\delta_m\) satisfies \(p_{\delta_m} = p(\pi_b)\). If such \(p_{\delta_m}\) does not exist, then \(\delta_m = 0\) or \(\delta_m = \delta\), depending on whether \(p(\pi_b) > p_0\) or \(p(\pi_b) < p_\delta\).

The mechanism is summarized as follows.

1. The bank announces the loan terms, specifically the collateral value function \(v(\beta)\). Entrepreneurs calculate the optimal borrowing and its associated utility;

2. Entrepreneurs and investors submit to the Walrasian auctioneer the part of their demands which generates more utility than from borrowing as in the previous step;

3. The auctioneer sets a price to clear the market for as many entrepreneurs as possible;

4. Entrepreneurs whose demands are not accepted in the market will borrow from the bank.

The following flowchart summarizes the sequence of actions for entrepreneurs at time 0.

Calculate the optimal borrowing from the bank \(\implies\) Submit demands in the market
\[\implies \begin{cases} \text{Transact in the market} & \text{if demands are cleared} \\ \text{Borrow from the bank} & \text{if demands are not cleared} \end{cases} \]
The equilibrium is defined as follows.

**Definition 2 (Bank and Market Equilibrium)** At time 0, the market is in equilibrium if the following conditions are satisfied:

1. Banks fully protect the loan loss by requiring sufficient collateral;
2. All agents make optimal decisions;
3. The market clears for \( x \).

**Proposition 3 (Nonempty Borrowing Equilibrium)** Given \( \Phi_m \), in the time 0 bank and market equilibrium, the measure of entrepreneurs borrowing from the bank is positive, that is, \( \delta_m < \delta \).

This proposition distinguishes the endogenous lending market from the restricted one, for borrowing activities do exist in equilibrium for the former.

If some of the entrepreneurs leave the market to borrow from the bank, there will be less trading in the market with investors. The collateralizable asset price is under less pressure from the selling. As a result, the price can be higher when the asset can be used as collateral, in line with the prediction by Geanakoplos (2003).

**Corollary 1 (Collateralizable Asset Price)** The price of an asset is higher when it can be used as collateral.

### 3 Implications for Asset Returns

Mayers (1972) and (1973) has extended the Capital Market Pricing Model by including a nonmarketable asset for each agent in the economy. He focuses on how
the expected returns of the marketable assets are affected by their correlations with nonmarketable assets. He has derived an asset pricing model in a linear form similar to the CAPM. Both the CAPM and Mayers’ extended model assume an exogenous lending market. In this section, by assuming independent correlation between the marketable and nonmarketable assets, we revisit the asset pricing model by endogenizing the lending market. Specifically, we link the lending market directly to the profitability of the private investment. The private investment is very similar to the nonmarketable assets in Mayers’ model, except that his nonmarketable assets pay a lump sum of money while the private investment is a production technology requiring input of cash in the beginning. Further assume all entrepreneurs possess the identical private investment. The goal here is to make bank lending less attractive so that the endogenous lending market can be distinguished from the unlimited one in which there’s no borrowing constraints. The identical private investment represents high systematic risk, suggesting these entrepreneurs are in the same industry or in the same region.

To be in line with the CAPM, assume both entrepreneurs and investors have mean-variance utility functions. The conventional CAPM considers two exogenous lending markets: agents can either borrow without constraints (unlimited lending market) or cannot borrow at all (restricted lending market). With the endogenous lending market, there are three patterns of the relationship between the expected returns of the collateralizable asset and the private investment. In the mean-variance economy with exogenous lending markets, only the mean and variance play a role. For the endogenous lending market, the probability distribution of the private investment also matters, especially the value of minimum return, because a zero VaR stipulates
that the bank considers the worst scenario. To demonstrate the difference among the three patterns, we construct a series of private investments with the same variance and increasing means from zero to infinity.

The difficulty is to show the difference between the endogenous and unlimited lending market. In the mean-variance economy, the collateralizable asset price is not affected by the variation of the private investments in the unlimited lending market. In the endogenous lending market, if bank lending is always a better way to finance than asset sale, there will be no asset sale or no trading in the market, leading to a price as if there is no private investment, the same result as the unlimited lending market. To demonstrate the difference, we show that asset sale does exist in certain equilibrium by using a particular form of the probability distribution for the series of private investments. With proposition 3, we show the existence of both bank lending and asset sale, distinguishing the endogenous bank lending model from the conventional exogenous restricted and unlimited lending markets.

Firstly, we derive an asset pricing model with an exogenous lending market. Secondly, we endogenize the lending market and compare the difference.

### 3.1 Exogenous Bank Lending

Assume all agents have the same mean-variance utility function with risk tolerance $\gamma$. In addition, assume they are all endowed with the same quantity of $x$, $c_i(0) = 1$ for $i \in [0, 1]$. Denote by $y_s$ the series of identical private investments for entrepreneurs, $s \in \left[-(2q - 1)t, \infty\right)$ in which $q$ is arbitrarily close to one and $t$ satisfies the following condition:
Condition 6: \( \frac{\delta}{1-\phi}(t-1) > 1. \)

For \( s \in [0, \infty) \), let \( y_s \) be a binomial random variable such that

(i) \( y_s(\omega_1) = s \) and \( y_s(\omega_2) = s + t \), and

(ii) \( \Pr(\omega_2) = q > \frac{1}{2} \).

And for \( s \in [-(2q-1)t, 0) \), let

(I) \( y_s(\omega_1) = 2qt - s \) and \( y_s(\omega_2) = (2q-1)t + s \).

All the private investments \( y_s \) have the same variance \( q(1-q)t^2 \) and an increasing mean from \( (1-q)t \) to \( \infty \). The goal is to prove for private investment \( y_0 \), some entrepreneurs finance the private investment \( y_0 \) by selling asset \( x \) in equilibrium.

For private investments, denote \( \mu_y(s) = Ey_s \) and \( \sigma_y^2 = \text{var}(y_s) \) where the variance is constant and does not change with \( s \). In addition, denote \( \mu_x = Ex \) and \( \sigma_x^2 = \text{var}(x) \) for the collateralizable asset \( x \). To highlight the role of using asset \( x \) solely to finance, assume entrepreneurs have no cash, \( e(1) = 0 \). The results in this section can be relaxed with a positive amount of endowed cash.

For entrepreneurs in \( [0, \delta] \), they maximize

\[
E(a_ix + b_i + c_iy_s) - \frac{1}{\gamma}\text{var}(a_ix + b_i + c_iy_s)
\]

subject to

(i) \( a_ip + b_i + c_i = e_i(0)p \)

(ii) \( a_i \geq 0 \) and \( c_i \geq 0 \),

where \( a_i \) is the \( x \) position, \( b_i \) is the cash position, \( c_i \) is the \( y_s \) position and \( p \) is the
market price of $x$. In the unlimited lending market, the cash holding $b_i$ can be either positive or negative, namely, there’s no restriction. In the restricted lending market, the cash holding $b \geq 0$. For investors in $(\delta, 1]$, they maximize the same objective function (11) without $y_s$.

The goal is to find the relationship between $\frac{\mu_x}{p}$ and $\mu_y(s)$, the expected returns of $x$ and $y_s$, respectively.

A. The unlimited Lending Market

The price for $x$ satisfies the first order condition

$$\frac{\mu_x - \frac{2\sigma^2}{\gamma}}{p} = 1.$$ (12)

The expected return for asset $x$, $\frac{\mu_x}{p}$, is a constant number, regardless of the change in $\mu_y(s)$. The relation between the two expected returns $\frac{\mu_x}{p}$ and $\mu_y(s)$ is shown in figure 3.

B. The Restricted Lending Market

Now consider an economy without lending. In order to raise more cash for private investment $y_s$, entrepreneurs have to sell $x$. The endowment arrangement satisfies the starting equilibrium condition 2 with price $p^* = \mu_x - \frac{2\sigma^2}{\gamma}$.

Additional conditions are needed to satisfy the mean-variance utility preference. Since asset $x$ generates a positive payoff, all agents prefer more to less, at least in the range of $[0, 1]$. This requirement gives the following condition.
Condition 7

\[
\frac{\gamma \mu_x}{2\sigma_x^2} \geq 1 \quad \text{for all } i \in [0, 1]
\]  

(13)

In equilibrium, the market price \( p \) is determined by the investors’ optimal holding of asset \( x \)

\[
\bar{a}_i = \frac{\gamma (\mu_x - p)}{2\sigma_x^2}, \quad i \in (\delta, 1].
\]

(14)

Denote by \( \bar{a}_i \) the optimal holding of asset \( x \) for entrepreneurs in \([0, \delta] \). By symmetry, each entrepreneur in equilibrium holds the same quantity of \( x \) and invests the same amount of cash in \( y_s \). If the investment in \( y_s \) is nonzero, the entrepreneur must have sold some \( x \) to investors to raise the needed cash, and hence holds less \( x \) than investors do. The marginal utility of holding \( x \) for entrepreneurs in \( i \in [0, \delta] \) is \( \mu_x - \frac{2\bar{a}_i \sigma_x^2}{\gamma} \), which is greater than \( \mu_x - \frac{2\bar{a}_j \sigma_x^2}{\gamma} = 1 \) for investors in \( j \in (\delta, 1] \) due to \( \bar{a}_i < \bar{a}_j \). Therefore, it’s optimal for entrepreneurs to hold no cash. For them, the marginal utility of \( x \) and \( y_s \) must be the same

\[
\frac{1}{p}(\mu_x - \frac{2\bar{a}_i}{\gamma} \sigma_x^2) = \mu_y(s) - \frac{2\bar{c}_i}{\gamma} \sigma_y^2.
\]

(15)

Solving it, this equation yields the optimal holdings of \( x \)

\[
\bar{a}_i = \frac{\frac{2}{\gamma} \sigma_y^2 p + \mu_x - p\mu_y(s)}{\frac{2}{\gamma}(\sigma_x^2 + p\sigma_y^2)}, \quad i \in [0, \delta].
\]

(16)

Given price \( p \), \( \bar{a}_i \) is seen to be a decreasing function of \( \mu_y(s) \). But the equilibrium price \( p \) of asset \( x \) is also a function of \( \mu_y(s) \). To consider the full effect of \( \mu_y(s) \) on the asset’s optimal holding \( \bar{a}_i \), take the derivative of \( \bar{a}_i \) with respect to \( p \) to obtain

\[
\frac{\frac{4}{\gamma} \sigma_y^2 \sigma_x^2 - \mu_x \sigma_y^2 - \mu_x \sigma_y^2}{(\frac{4}{\gamma} \sigma_x^2 + p \sigma_y^2)^2},
\]

which is less than zero according to equation (13). For entrepreneurs, \( \bar{a}_i \) is therefore a decreasing function of both \( p \) and \( \mu_y \). Seen from equation (14), the
investors’ optimal holding of asset $x$, $\bar{a}_i$, is a decreasing function of $p$. It follows that the equilibrium price $p$ is a decreasing function of $\mu_y(s)$. Otherwise, the optimal holdings of $x$ for all the agents $\bar{a}$ decrease as $\mu_y(s)$ increases and the market cannot clear. The relationship between the two expected returns of $x$ and $y$ is shown in figure 4.

**Proposition 4** In equilibrium, the price of asset $x$ is a decreasing function of $\mu_y$, the expected return of asset $y$.

### 3.2 Endogenous Bank Lending

Intuitively, when $\mu_y(s)$ is low, the expected return from $x$ should also be low, for entrepreneurs have little incentive to sell $x$ in exchange for cash to invest in $y$. On the other hand, as $\mu(s)$ increases, entrepreneurs become willing to sell more $x$ to raise cash, which could potentially push down the price. Meanwhile, the loan from the bank becomes increasingly attractive as the collateral value appreciates with the increasing return of $y$. As a result, more entrepreneurs leave the market for the bank. Under less selling pressures, the market price of $x$ remains high, causing a low expected return. To sum up, the price of $x$ is the same as in equation (12) when $\mu_y(s)$ is extremely low or high, that is, when $s = -(2q - 1)t$ or $s > 1$, respectively. The remaining riddle is for the values of $\mu_y(s)$ in the middle range. In the rest of this section, we show that the price of asset $x$ is indeed lower than equation (12) for private investment $y_0$, distinguishing the endogenous lending market from the unlimited one.

Now assume $\beta v(\beta)$ is an increasing function of $\beta$ for any collateral value function $v$. Borrowers fail to repay the loan only when $\omega_1$ happens, that is, $y_0(\omega_1) = 0$. In this
scenario, no entrepreneurs own any cash after suffering a bad return on the private investment. The bank has to liquidate the collateral, and hence the liquidation price is $\mu_x - \frac{\delta + 1 - \delta}{1 - \delta} \frac{2\sigma^2}{\gamma}$ for an amount of $\beta \delta$ units of $x$ to be liquidated. Take the derivative of $\beta v(\beta) = \beta (\mu_x - \frac{\delta + 1 - \delta}{1 - \delta} \frac{2\sigma^2}{\gamma})$ with respect to $\beta$ and let it be greater than zero to obtain the following condition.

**Condition 8 (Monotonicity)** $p^* - \frac{4\delta\sigma^2}{(1-\delta)\gamma} > 0$.

Under condition 8 and 6, the participants in the market will be nonempty.

**Proposition 5 (Nonempty Market)** *For private investment $y_0$, some entrepreneurs trade in the market in equilibrium for a certain value of the probability $q$.***

**Corollary 2** *From both propositions 5 and 3, the endogenous lending market is indeed different from the two commonly used exogenous lending markets: the restricted and unlimited lending markets. For a certain return $y$ of the private project, in the endogenous lending market, ex ante homogeneous entrepreneurs choose different optimal strategies, whereas in the two endogenous lending markets, they choose the same optimal one.*

For such a series of private investments $y_s$ in the proposition, the relationship between the expected returns of $x$ and $y_s$ is shown in figure 5.

### 3.3 Summary

The private investments have both the substitution and wealth effects on the collateralizable asset's return, because on the one hand, they compete with the collateral-
izable assets for the entrepreneurs’ limited wealth, and on the other hand, they define
the entrepreneurs’ borrowing limits. According to the starting equilibrium condition,
both entrepreneurs and investors already hold the optimal wealth portfolios without
the private investments. The price of the collateralizable asset declines only when en-
trepreneurs sell it to investors in order to raise cash. The more profitable the private
investments, the more entrepreneurs want to sell the collateralizable asset in order to
invest. This is known as the substitution effect. The wealth effect derives from the
way the endogenous lending market works. The borrowing capacity for entrepreneurs
is the collateralizable asset’s value used as collateral, that is, the value in the future
after entrepreneurs receive the payoff from their private investments. As these private
investments become more profitable, entrepreneurs’ wealth increases. This increased
wealth bids up the collateral value and hence the borrowing capacity. This in turn re-
duces the need to raise cash by selling the collateralizable asset. As a result, the price
of the collateralizable asset remains the same as when there is no private investment.

4 Banks Facilitate Diversification

So far in this economy, each entrepreneur has a given private investment. They
make decisions on how to allocate capital between the collateralizable asset $x$ and
their private investments. In this section, we study how entrepreneurs choose among
different private investments in equilibrium. Traditional capital budgeting theories
study it mainly from the firms’ (entrepreneurs’) perspective and is confined to a
partial equilibrium. In this section, however, we focus on the effect of bank loans on
the entrepreneurs’ choices. As the lending market is endogenously determined in the
aggregate economy, all firms’ decisions are made in a general equilibrium.

A. An example with two choices for entrepreneurs

To keep the model solvable without losing insights, assume two perfectly hedgeable securities \( y_A \) and \( y_B \) with two states \( \omega_1 \) and \( \omega_2 \) such that

\[
\begin{align*}
y_A(\omega_1) &= \rho + \epsilon, \quad y_A(\omega_2) = 0, \\
y_B(\omega_1) &= 0, \quad y_B(\omega_2) = \rho, \\
Pr(\omega_1) = Pr(\omega_2) &= \frac{1}{2}
\end{align*}
\]

where \( \epsilon > 0 \) is an arbitrarily small number and \( \rho > 2 \). In each state, there’s only one security generating a positive return. Furthermore, assume both \( y_A \) and \( y_B \) are independent of \( x \). Each entrepreneur can choose either \( y_A \) or \( y_B \) as his personal investment at time 0 to maximize

\[
Eu_i(a_i x + b_i + c_i y_i)
\]

subject to

(i) \( a_i p + b_i + c_i = c_i(0)p + c_i(1) \)

(ii) \( y_i \in \{y_A, y_B\} \).

For convenience, define a type-A(type-B) entrepreneur as one choosing \( y_A(y_B) \). The objective function for investors are the same as before without \( \{y_A, y_B\} \).

In both scenarios: without bank lending or for exogenous bank lending, both type-A and type-B entrepreneurs obtain the same loan contracts. As a result, they always
prefer \( y_A \) to \( y_B \) when maximizing equation (17). In equilibrium, all entrepreneurs select \( y_A \).

With endogenous bank lending, however, bank loans are no longer the same for both the type-A and type-B entrepreneurs. In other words, when type-A and type-B entrepreneurs use the same quantity of asset \( x \) as collateral to borrow, the collateral value, and hence the amount of the loan, will be different. The greater the number of the same type of entrepreneurs in the economy, the less each can borrow, because banks liquidate the collateral at a more distressed time, in the same spirit as Shleifer and Vishny (1992). In the extreme case when all entrepreneurs choose \( y_A \), the collateral value for a potential type-B entrepreneur is at the largest. The timing when the type-B entrepreneur defaults is associated with a good return for all type-A entrepreneurs. When compensated by the better loan from banks, some entrepreneurs are expected to switch from \( y_A \) to \( y_B \). Overall, banks prefer those entrepreneurs whose wealth is negatively correlated with the aggregate economy. The formal proof is in the following proposition.

**Proposition 6 (Diversification)** The measure of the set of entrepreneurs choosing the inferior investment \( y_B \) is positive if \( \epsilon \) is small enough.

In the proof of the proposition, we show that the inferior investment can only be financed by borrowing. If an entrepreneur does not borrow, he must choose the more profitable investment. This prediction is in line with the existing empirical findings on the negative correlation between leverage and profitability.

**Lemma 1 (Subsidized Loan)** Leverage is essential for less profitable investments.
4.1 Summary

Our model predicts that banks can facilitate diversification among the entrepreneurs. Usually, banks minimize credit risk by lending to a diversified group of entrepreneurs. With collateral, it is easier for banks to achieve this, because the collateral value is already based on the aggregate economy. The more diversified the aggregate economy, the higher the collateral value (proposition 2). In a sense, even if a bank only lends to one entrepreneur, it faces the same minimum risk as when it lends to many. Therefore, collateral lending makes it easier for banks to diversify their risk.

5 Conclusion

In this paper, we study an endogenous lending market that considers the interaction among borrowers. This interaction effect is absent in much of the existing financial literature. In this endogenous lending market, we are able to show both the wealth and substitution effects in the relationship between returns of two assets. The wealth effect, however, does not exist in the conventional asset pricing literature that assumes an exogenous lending market.

In addition, we also examine capital budgeting issues with a bank-firm joint analysis by granting two investment options to entrepreneurs. Traditional capital budgeting theories often rank investments according to profitability. Using a bank-firm joint analysis, we show that profitability is no longer the sole criterion. Since banks manage a portfolio of credit risks, the correlation between the two investments also plays a role. Less profitable investments could exist in equilibrium due to subsidized
loans from banks, if these investments are negatively correlated with the aggregate economy. Moreover, the fact that they must rely on loans is in line with empirical findings about the negative correlation between leverage and profitability. Indeed, more profitable investments can be executed without leverage in the equilibrium.

With the assumption of perfect information in the model, we mainly focus on the benefits of using collateral to borrow, such as the reduced asset price volatility and diversification in an economy. Xu(2011) indicates, however, that the use of collateral can cause two severe consequences: 1) market instability and 2) contagion in an imperfect information environment where banks have little information about entrepreneurs who are not their customers. This study suggests that high quality information is essential in an economy that uses collateral to borrow, more so than one that does not. As we mentioned in the introduction, the use of collateral connects each individual’s borrowing capacity to the aggregate economy, whereas, without collateral, this borrowing capacity is determined at the individual level. In one sense, the use of collateral helps create a more closely-linked economy. To fully exploit the advantage of such an economy, high quality information is essential.

References


6 Appendix

Proof of proposition 1

Proof. Denote by $\beta$ and $c$ the amount of asset $x$ an entrepreneur uses as collateral to borrow and the cash position, respectively. Given two returns $y_2 > y_1$, it suffices to show that if the entrepreneur is willing to repay fully the loan with return $y_1$, he must be willing to do so with return $y_2$. Denote by $u_i = Eu(e(0)x + (e(1) - c + \beta v(\beta))y_i - \beta v(\beta))$. With return $y_2$, the ability to serve the loan is not an issue for he has more cash now. Denote by $A(u, a)$ the minimal amount of cash together with $a$ units of $x$ to generate the utility $u$. Since the entrepreneur is willing to repay the loan with return $y_1$, it immediately follows that $A(u_1, e(0) - \beta) > A(u_1, e(0)) > \beta v(\beta)$. It’s sufficient to show $A(u_2, e(0) - \beta) - A(u_2, e(0)) > A(u_1, e(0) - \beta) - A(u_1, e(0))$. Denote by $P(a, b)$ a price function such that the entrepreneur is optimal to hold $a$ units of $x$ and $b$ units of cash. According to the first order condition $\frac{\partial A(u, a)}{\partial a} = -P(a, A(u, a))$, it holds that

$$\int_{e(0) - \beta}^{e(0)} P(a, A(u_1, a)) da = A(u_1, e(0) - \beta) - A(u_1, e(0)) > \beta v(\beta)$$

(18)

If $P(a, A(u_2, a)) < P(a, A(u_1, a))$ holds, then it’s true for the claim that the entrepreneurs is willing to repay the loan with return $y_2$, according to equation (18). From $A(u_2, a) > A(u_1, a)$ and condition 4, it must be $a = a^*(P(a, A(u_2, a)), a, A(u_2, a)) > a^*(P(a, A(u_2, a)), a, A(u_1, a))$. Since $a^*(P(a, A(u_1, a)), a, A(u_1, a)) = a$, it must follow $P(a, A(u_1, a)) < P(a, A(u_2, a))$ from condition 5.

$\blacksquare$
Proof of proposition 3

Proof. Denote by $\pi_{\delta_m}$ the utility for an entrepreneur in the market with entrepreneurs in $[0, \delta_m]$ and investors in $(\delta_m, \delta]$. And denote by $\pi_{b\delta_m}$ the maximum utility for a borrower when the bank offers a loan with collateral value function $v_{\delta_m}(\beta, \tilde{y})$ based on the guess that entrepreneurs $[0, \delta_m]$ transact in the market. In equilibrium, it must be $\pi_{b\delta_m} = \pi_{\delta_m}$.

Claim that $\pi_{\delta_m}$ is a decreasing function of $\delta_m$. As there are more entrepreneurs in the market, each sells the same quantity of $x$ for less price and hence receives a lower utility.

Consider $\delta_m = 0$. If $\pi_{b0} \geq \pi_0$, all entrepreneurs are optimal to borrow from the bank, and the proposition follows. Assume otherwise $\pi_{b0} < \pi_0$.

Now compare the two utilities at $\delta_m = \delta$ and prove $\pi_{b\delta} \geq \pi_{\delta}$. Assume otherwise $\pi_{b\delta} < \pi_{\delta}$ and it’s an equilibrium for the market with all entrepreneurs. Consider a small fraction of entrepreneurs $(\delta - \epsilon, \delta]$ now switch to borrow from the bank using $a_\delta$ units of $x$ as collateral to borrow, the same quantity sold in the market. The value of the collateral $v_{\delta}(a_\delta)$ is the price the bank receives when selling $\epsilon a_\delta \Pr(D)$ units of $x$ in the market. According to the mechanism, the bank will quote a asking price. For entrepreneurs in the market who obtain a return greater than 1, they are willing to buy $e(0) - a_{\delta-\epsilon}$ units of $x$ at price $p^*$. As long as $\epsilon$ is small enough so as to $(e(0) - a_{\delta-\epsilon})(\delta - \epsilon) \Pr(y \geq 1) \geq \epsilon a_\delta \Pr(D)$, the bank can sell the collateral for at least $p^*$. Therefore, the borrowers can obtain a larger sum of cash by using $a_\delta$ units of $x$ as collateral. Since using $a_\delta$ units of $x$ as collateral to borrow is not necessarily the best strategy for entrepreneurs $(\delta - \epsilon, \delta]$, they can achieve even higher utility with
banks. Thus it won’t be an equilibrium for the market with all entrepreneurs. It immediately follows $\pi_{\delta_0} \pi_{\delta}$.

Now that both $\pi_{\delta_0} < \pi_0$ and $\pi_{\delta_0} \pi_{\delta}$ hold, there must exist at least one $\delta_m$ such that $\pi_{\delta_0} \pi_{\delta_m} = \pi_{\delta_m}$ from continuity. Denote by $\Delta = \{ \delta_m | \pi_{\delta_0} \pi_{\delta_m} = \pi_{\delta_m} \}$ a set consisting of all such $\delta_m$. By continuity again, $\Delta$ is a closed set. Then there’s only one equilibrium with $\delta^* = \min \Delta$ that generates the highest utility among all $\Delta$, for a bank can always offer such a loan with collateral value function $v_{\delta^*_m}(\beta)$ to attract all the potential borrowers. The proof is complete. ■

Proof of proposition 5

Proof. Assume otherwise all entrepreneurs prefer to borrow from the bank. Denote by $\beta$ the quantity of $x$ in equilibrium used as collateral to borrow. Since all entrepreneurs default at the same time, the collateral value $v(\beta) = \mu_x - \frac{\beta^2 + 1 - \delta}{1 - \delta} \frac{2 a^2}{\gamma}$ is the price when the bank sells total $\beta \delta$ units of $x$ to investors. Denote by $U_b$ the utility for the entrepreneurs borrowing from the bank. It’s important to note that the collateral value $v(\beta)$ won’t change as the probability of obtaining good return $q$ increases, as long as $q < 1$. Therefore it follows that $\lim_{q \to 1} U_b = \mu_x + \beta v(\beta)(t - 1) - \frac{a^2}{\gamma} \leq \mu_x + v(1)(t - 1) - \frac{a^2}{\gamma}$.

If a very small fraction of entrepreneurs $[0, \epsilon_1]$ switch to transact in the market, the market equilibrium price $p_{\epsilon_1}$ satisfies $\lim_{\epsilon_1 \to 0} = p^*$. Denote by $U_m(\epsilon_1)$ the utility for entrepreneurs in the market. Then it follows $\lim_{q \to 1} \lim_{\epsilon_1 \to 0} U_m(\epsilon_1) \geq a \mu_x + (1 - a) p^* t - \frac{a^2}{\gamma}$, for any $a \in [0, 1]$. Particularly, let $a = 0$ and obtain $\lim_{q \to 1} \lim_{\epsilon_1 \to 0} U_m(\epsilon_1) \geq p^* t$.

According to condition 6, it holds that $p^* t > \mu_x + v(1)(t - 1) - \frac{a^2}{\gamma}$. Therefore
by continuity, for a $q$ arbitrarily close to 1, it’s optimal for a small fraction of entrepreneurs to transact in the market. ■

**Proof of proposition 6**

**Proof.** Assume otherwise no entrepreneurs choose $y_B$. In equilibrium, denote by $\delta_m$ such that entrepreneurs in $[0, \delta_m]$ transact in the market and those in $(\delta_m, \delta]$ borrow from banks. The proof is for $\delta_m < \delta$ only. A similar argument applies to $\delta_m = \delta$.

Denote by $L(z)$ a price enabling investors to buy $z$ units of $x$. It immediately follows that the market equilibrium price $p_{\beta_m} = L(a_{\delta_m} \delta_m)$ and $L(0) = p^*$. The collateral value function for A-type entrepreneurs is $v_A(\beta) = L(a_{\delta_m} \delta_m + \beta(\delta - \delta_m))$. This is because banks can only sell collateral to investors when a type-A entrepreneur defaults. Given there’s no B-type entrepreneurs, the collateral value function for them is $v_B(\beta) \geq p^* - \epsilon_1$ for an arbitrarily small number $\epsilon_1 > 0$, because the bank only liquidate $x$ when A-type entrepreneurs obtain good return $y_A = \rho$. It must hold that $v_B(\beta) > v_A(\beta)$.

Now, if a fraction of borrowers switch to security $y_B$ and use $\beta_{\delta_m}$ units of $x$ as collateral for a loan, he can obtain $\beta_{\delta_m} v_B(\beta_m)$ cash from the bank. Let $\epsilon$ be small enough so that $\beta_{\delta_m} v_B(\beta_m) > \beta_{\delta_m} v_A(\beta_m)$. Then the borrower achieves a higher utility than before. Yet using $\beta_{\delta_m}$ is not necessarily the optimal strategy with collateral value function $v_B(\beta)$. The maximum utility with $v_B(\beta)$ is greater than that with $v_A(\beta)$. Therefore, all borrowers should switch to security $y_B$ and the equilibrium breaks down. The claim that no entrepreneurs choose security $y_B$ is false and the proposition follows. ■
7 Figures

Figure 2: An example of demands for investors and entrepreneurs

Figure 3: The relationship between the expected returns of x and y in an exogenous lending market
Figure 4: The relationship between the expected returns of x and y in an exogenous lending market

Figure 5: The relationship between the expected returns of x and y in an endogenous lending market